Large Cardinals
Lent Term 2023
Part III of the Mathematical Tripos
University of Cambridge
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Example Sheet #3

Examples Classes.
#1: Friday 10 February 2023, 1:30–3:30pm, **MR5**.
#2: Friday 3 March 2023, 1:30–3:30pm, **MR5**.
#3: Thursday 16 March 2023, 3:30–5:30pm, **MR3**.

Presentation. Two of the examples are designed to be a Presentation Example (marked on the sheet). We encourage all students to meet in pairs, work together on these examples, and prepare a short presentation of their solutions that can be given on the blackboard in **MR4** during the examples class. The discussion during your meeting should be both about the mathematical content and about the preparation of the presentation.

Marking. You can submit all of your work to Ioannis Eleftheriadis (`ie257`) as a *single pdf file* by e-mail or hand it to him on paper during the examples class. Please submit all work before the start of the examples class. Work that is submitted at least 24 hours before the examples class could already be marked and returned during the examples class. We cannot guarantee that all work will be marked, but we shall endeavour to mark at least two examples per submission. Model solutions will be provided on the moodle page of the course.

For Examples (31) to (37), we assume that $\kappa < \lambda$ are measurable and inaccessible, respectively, that M is the Mostowski collapse of the ultrapower of $\mathbf{V}_\lambda$ modulo U and that $j : \mathbf{V}_\lambda \to M$ is the ultrapower embedding. We use the notation from the lectures.

(31) In Lecture XI (pages 5 & 6), we showed that $\kappa \leq (\mathrm{id}) < j(\kappa)$. Give concrete functions $f : \kappa \to \kappa$ such that $(f) = (\mathrm{id}) + 1$, $(f) = (\mathrm{id}) + \omega_1$, $(f) = (\mathrm{id}) \cdot 2$. Fix $\xi < \kappa$ and consider the function $f(\alpha) := \xi$ if α is even and $f(\alpha) := \alpha$ if α is odd. What can we say about the relation between (id) and (f)?

[*Remark.* As usual, an ordinal α is even if it is of the form $\lambda + 2n$ where λ is a limit ordinal and n is a natural number.]

(32) Show that if $U \in M$, then (in M) there is a surjection from κ^κ onto $j(\kappa)$. Deduce that $U \notin M$.

(33) Presentation Example. Let μ be a cardinal. We say that a set $X \subseteq \mathbf{V}_\lambda$ is *closed under μ-sequences* if $X^\mu \subseteq X$. Show that M is closed under κ-sequences, but not under κ^+-sequences.

[*Hint.* For the second claim, show that $s : \kappa^+ \to M : \alpha \mapsto j(\alpha)$ is not an element of M.]

(34) Let U be a κ-complete nonprincipal ultrafilter on κ. Show that if $\{\alpha+1\,;\,\alpha<\kappa\}\in U$, then U cannot be normal. Use this to show that if κ is measurable, then there are κ-complete nonprincipal ultrafilters on κ that are not normal.

[*Hint.* Use the fact that $\{\alpha+1\,;\,\alpha<\kappa\}$ is in bijection with κ and Example (20).]

(35) In Lecture XIII, we claimed that U is normal if and only if $(\mathrm{id})=\kappa$ and showed the direction from left to right. Prove the other direction.

(36) Let $(f)=\kappa$ for some $f:\kappa\to\kappa$. Show that $\{X\subseteq\kappa\,;\,f^{-1}[X]\in U\}$ is a normal ultrafilter on κ.

(37) Presentation Example. Assume that $M\models$"κ is measurable" witnessed by an ultrafilter $(g)\in M$ and $(f)=\kappa$. Use this to show directly (without using a reflection argument) that there are κ many measurable cardinals below κ.

(38) Let Φ and Ψ be sentences of the language of set theory. Show that if $\mathsf{ZFC}+\Phi\leq_{\mathrm{Cons}}\mathsf{ZFC}+\Psi$, then $\mathsf{ZFC}+\Phi\vee\Psi\equiv_{\mathrm{Cons}}\mathsf{ZFC}+\Phi$.

(39) Let $\Xi(\kappa)$ be "κ is inaccessible and if κ is weakly compact, then there is a $\lambda>\kappa$ that is inaccessible". Show that $\mathsf{ZFC}+\Xi\mathsf{C}\equiv_{\mathrm{Cons}}\mathsf{ZFC}+\mathsf{IC}$. Use this to argue that the following statement is in general false for sentences Φ and Ψ in the language of set theory: if $\mathsf{ZFC}+\Phi\leq_{\mathrm{Cons}}\mathsf{ZFC}+\Psi$, then $\mathsf{ZFC}+\Phi\wedge\Psi\equiv_{\mathrm{Cons}}\mathsf{ZFC}+\Psi$.

(40) Let $\infty\mathsf{IC}$ be the statement "there are unboundedly many inaccessible cardinals". Under the assumption of $\infty\mathsf{IC}$, let λ_α be the αth inaccessible cardinal. Show that it is not possible to prove in $\mathsf{ZFC}+\infty\mathsf{IC}$ that the operation $\alpha\mapsto\lambda_\alpha$ has a fixed point, i.e., some $\kappa=\lambda_\kappa$. This must mean that the operation is in general not a normal ordinal operation. Use this to show that the consistency strength of the statement "there is a 1-inaccessible cardinal" (cf. Example (24)) is strictly stronger than $\infty\mathsf{IC}$.

(41) We define by recursion: κ is 0-*measurable* if it is measurable; κ is $\alpha+1$-*measurable* if it α-measurable and there are unboundedly many α-measurables below κ; for a limit ordinal $\lambda\leq\kappa$, κ is λ-*measurable* if it is ξ-measurable for all $\xi<\lambda$.

Show that if κ is surviving, it is κ-measurable.

(42) A cardinal κ is called *strongly compact* if for any set S of nonlogical symbols (with no size restriction), any $\mathcal{L}_{\kappa\kappa}$-language L_S, and any $\Phi\subseteq L_S$, if Φ is κ-satisfiable, then Φ is satisfiable. Prove that if κ is strongly compact and F is any κ-complete filter on any set X, then there is a κ-complete ultrafilter $U\supseteq F$.

(43) Prove that every strongly compact cardinal is measurable.