Prof. Dr. B. Löwe, I. Eleftheriadis

# Example Sheet #2

**Examples Classes.**
#1: Friday 10 February 2023, 1:30–3:30pm, **MR5**.
#2: Friday 3 March 2023, 1:30–3:30pm, **MR5**.
#3: Thursday 16 March 2023, 3:30–5:30pm, **MR3**.

**Presentation.** Two of the examples are designed to be a Presentation Example (marked on the sheet). We encourage all students to meet in pairs, work together on these examples, and prepare a short presentation of their solutions that can be given on the blackboard in **MR4** during the examples class. The discussion during your meeting should be both about the mathematical content and about the preparation of the presentation.

**Marking.** You can submit all of your work to Ioannis Eleftheriadis (`ie257`) as a *single pdf file* by e-mail or hand it to him on paper during the examples class. Please submit all work before the start of the examples class. Work that is submitted at least 24 hours before the examples class could already be marked and returned during the examples class. We cannot guarantee that all work will be marked, but we shall endeavour to mark at least two examples per submission. Model solutions will be provided on the moodle page of the course.

(15) A set model $(M, E) \models \mathsf{ZFC}$ is called an *$\omega$-model* if there is an isomorphism between $(\{x \in M\,;\, M \models \text{“}x \text{ is a natural number”}\}, E)$ and $(\omega, \in)$. Assume that $\mathsf{ZFC}$ is consistent and use the compactness theorem to construct a model of $\mathsf{ZFC}$ that is not an $\omega$-model.

(16) We write $\mathsf{ZFC}_\omega$ for $\mathsf{ZFC}$+“there is an $\omega$-model of $\mathsf{ZFC}$”. Work in $\mathsf{ZFC}_\omega$ and let $M$ be an $\omega$-model of $\mathsf{ZFC}$; without loss of generality, we can assume that $\omega \subseteq M$. We encode formulas of first-order logic by natural numbers, writing $\ulcorner\varphi\urcorner$ for the number coding $\varphi$. Let $\Phi$ be a set of first-order sentences such that $\Phi$ exists in $M$, i.e., there is some $x \in M$ such that $\varphi \in \Phi$ if and only if $M \models \ulcorner\varphi\urcorner \in x$. Show that $\Phi$ is consistent if and only if $M \models$ “$\Phi$ is consistent”. Deduce that if $\mathsf{ZFC} + \mathrm{Cons}(\mathsf{ZFC})$ is consistent, then $\mathsf{ZFC} + \mathrm{Cons}(\mathsf{ZFC}) <_{\mathrm{Cons}} \mathsf{ZFC}_\omega$.

(17) A set model $(M, E)$ of $\mathsf{ZFC}$ is called a *$\beta$-model* if the relation $E$ is wellfounded. We write $\mathsf{ZFC}_\beta$ for $\mathsf{ZFC}$+“there is a $\beta$-model of $\mathsf{ZFC}$”. In our *Brief digression on nontransitive models* in Lecture V (pages 4–6), we sketched a construction of a countable transitive model of $\mathsf{ZFC}$ (this construction implicitly used the assumption $\mathsf{ZFC}_\beta$). Go through the details of this argument to check that you fully understand it; in particular, understand in which steps the assumption $\mathsf{ZFC}_\beta$ is needed.

(18) Let $U$ be an ultrafilter on $\kappa$. Show that $U$ is $\lambda$-complete if and only if for each $\gamma < \lambda$ and $\{A_\alpha\,;\, \alpha < \gamma\} \subseteq U$, we have that $\bigcap_{\alpha<\gamma} A_\alpha \neq \varnothing$.

(19) Using the Axiom of Choice, show that every filter can be extended to an ultrafilter.

(20) If $C$ is a set of subsets of $Z$, we say that $D$ is the *collection generated by* $C$ if $D$ is minimal such that $C \subseteq D \subseteq \wp(Z)$ and $D$ is closed under finite intersections and supersets.

Let $X$ and $Y$ be sets, $f : X \to Y$ a function, $F$ a filter on $X$ and $G$ a filter on $Y$. Let $f_*F$ be the collection generated by $\{f[A]\,;\, A \in F\}$ (called the *pushout of* $F$) and $f^*G$ be the collection generated by $\{f^{-1}[B]\,;\, B \in G\}$ (called the *pullback of* $G$).

(a) Under which conditions on $f$ are $f_*F$ or $f^*G$ filters?

(b) Under which conditions on $f$ are $\{f[A]\,;\,A \in F\}$ or $\{f^{-1}[B]\,;\,B \in G\}$ filters?

(c) If $F$ or $G$ are ultrafilters, are $f_*F$ or $f^*G$?

(d) If $F$ or $G$ are $\kappa$-complete, are $f_*F$ or $f^*G$?

(e) If $F$ or $G$ are nonprincipal, are $f_*F$ or $f^*G$?

(21) Presentation Example. A cardinal $\kappa$ is called an *Ulam cardinal* if there is an $\aleph_1$-complete non-principal ultrafilter on $\kappa$. Show that the smallest Ulam cardinal is a measurable cardinal.

(22) Find an $\mathcal{L}_{\omega_1\omega_1}$ formula that characterises the $\omega$-models of ZFC.

(23) Check the $\exists^\alpha$ case in the inductive proof of Łoś's Theorem for $\mathcal{L}_{\kappa\kappa}$ languages (cf. Lecture VIII, page 6).

(24) We define by recursion: $\kappa$ is 0-*inaccessible* if it is inaccessible; $\kappa$ is $\alpha+1$-*inaccessible* if it $\alpha$-inaccessible and there are unboundedly many $\alpha$-inaccessibles below $\kappa$; for a limit ordinal $\lambda \leq \kappa$, $\kappa$ is $\lambda$-*inaccessible* if it is $\xi$-inaccessible for all $\xi < \lambda$.

Show: if $\kappa$ is weakly compact, it is $\kappa$-inaccessible.

(25) Show that the Erdős arrow notation is stable under increasing numbers on the left hand side of the arrow and decreasing numbers on the right hand side of the arrow. I.e., if $\kappa \to (\lambda)^m_\mu$ and $\kappa' \geq \kappa$, $\lambda' \leq \lambda$, $\mu' \leq \mu$, and $m' \leq m$, then $\kappa' \to (\lambda')^{m'}_{\mu'}$.

(26) Let $\kappa$ be regular and $\lambda < \kappa$. Let $2^\lambda := \{f\,;\,f : \lambda \to \{0,1\}\}$ be ordered lexicographically by $f <_{\text{lex}} g$ if $f(\alpha) = 0$ and $g(\alpha) = 1$ if $\alpha$ is the least ordinal where $f$ and $g$ differ. Show that $(2^\lambda, <_{\text{lex}})$ has no strictly increasing or decreasing sequences of length $\kappa$.

(27) Let $U$ be a ultrafilter on $\kappa$ such that all elements of $U$ have cardinality $\kappa$. Show that if $U$ is normal, then $U$ is $\kappa$-complete.

(28) Let $\kappa$ be regular. A set $A \subseteq \kappa$ is *closed* if for each limit ordinal $\lambda < \kappa$, if $A \cap \lambda$ is unbounded in $\lambda$, then $\lambda \in A$. A set $C$ is called a *club set* (for "closed unbounded") if it is closed and unbounded. Define

$$\mathcal{C} := \{A \subseteq \kappa\,;\,\text{there is a club set } C \subseteq A\}.$$

Show that $\mathcal{C}$ is a $\kappa$-complete and normal filter on $\kappa$.

(29) Let $F$ be a filter on a cardinal $\kappa$. Say that for $X \subseteq \kappa$, a function $f : X \to \kappa$ is called *regressive* if $f(\alpha) < \alpha$ for all $0 \neq \alpha \in X$. A set $S$ is called $F$-*stationary* if for all $X \in F$, we have that $X \cap S \neq \varnothing$. Prove that the following statements are equivalent for a filter $F$.

(i) The filter $F$ is closed under diagonal intersections.

(ii) For any $F$-stationary set $S$ and any regressive $f : X \to \kappa$, there is an $\alpha < \kappa$ such that $f^{-1}(\{\alpha\})$ is $F$-stationary.

(30) Presentation Example. Let $\kappa < \lambda$ such that $\kappa$ is measurable with $\kappa$-complete ultrafilter $U$ and $\lambda$ inaccessible and let $M$ be the Mostowski collapse of the ultrapower $(\mathbf{V}_\lambda)^\kappa/U$. Show that for each $x \in M$, we have that $|\text{tcl}(x)| < \lambda$. Deduce that $M \subseteq \mathbf{V}_\lambda$.

[*Hint.* Consider Examples (5) & (6) from Example Sheet # 1.]