Large Cardinals

REVISION CLASS
25 MAY 2023

SELF ASSESSMENT

A Revision

LIST ALL THAT APPLY

1 Own lecture notes
2 Lecture pdfs
3 Minter notes on moodle
4 Example sheets
5 Past exam (2022)
6 Recommended books: Jech + Kanamori
7 Other resources

B Exam training

LIST ONE ANSWER

1 Real-time closed book under exam conditions DONE
2 Real-time closed book under exam conditions PLANNED
3 No plans

C Confidence

LIST ONE ANSWER

1 Not confident
2 Reasonable confident
3 Confident
4 Extremely confident

RUBRIC

MAT3
MATHEMATICAL TRIPOS Part III

Tuesday, 6 June, 2023 9:00 am to 11:00 am

PAPER 116

LARGE CARDINALS

Before you begin please read these instructions carefully

Candidates have TWO HOURS to complete the written examination.

Attempt **ALL** questions.
There are **THREE** questions in total.
Question 1 carries 40 marks; $\underbrace{\text{questions 2 and 3 carry 30 marks each.}}_{\text{"Problem questions"}}$

Bookwork question

FROM MOODLE

Lectures

First Lecture. *Monday 23 January 2023*. Large cardinal axioms: their goal in light of the incompleteness phenomenon; general idea: generalising largeness properties of ω to uncountable cardinals. Informal description of large cardinal axioms. Two non-examples: sizes of power sets violating GCH (existence not provable in ZFC, but not necessarily large) and fixed points of the aleph function (very large, but existence provable in ZFC). Reminder: proof that arbitrarily large fixed points of normal ordinal operations exist. Largeness properties of ω: limit cardinal, strong limit cardinal, regular. (Remark: smallest aleph fixed point has cofinality $\aleph_0$ and is thus singular.) Inaccessible cardinals. Inaccessible cardinals are aleph fixed points. Lecture Notes.

Second Lecture. *Wednesday 25 January 2023*. Using Gödel's Second Incompleteness Theorem to prove that something cannot be proved by ZFC. Von Neumann hierarchy and valid axioms of set theory in von Neumann ranks. Second order replacement. If κ is inaccessible, then $\mathbf{V}_\kappa$ satisfies second order replacement. Absoluteness of inaccessibility for von Neumann ranks. Second proof that ZFC does not prove IC, not using Gödel's Second Incompleteness Theorem: concrete construction of models of $\mathrm{ZFC} + \neg\mathrm{IC}$. Lecture Notes.

(Lecture V)

Third Lecture. *Monday 30 January 2023*. The consistency strength hierarchy: definitions, maximal element, historical connection to Hilbert's finitist program. Consistent theories T such that $T + \mathrm{Cons}(T)$ is inconsistent. The property of ω-consistency (no precise definition). Strictly increasing sequences of length ω under assumption of ω-consistency. Strictly increasing sequences of length ω under assumption of $\mathrm{ZFC} + \mathrm{IC}$. Definition of worldly cardinals. Lecture Notes.

Fourth Lecture. *Wednesday 1 February 2023*. If $\mathbf{V}_\kappa \models \mathrm{ZFC}$, then κ is a cardinal. In this case, κ is a limit cardinal and an aleph fixed point (no proof, cf. Example Sheet #1). Basic model-theoretic concepts: elementary equivalence, elementary substructure, elementary embedding. Basic model-theoretic results: Tarski-Vaught Test, elementary chains, Tarski's Chain Lemma. The set of worldly cardinals below an inaccessible cardinal is unbounded. There are worldly cardinals of all cofinalities below κ among them. Lecture Notes.

Fifth Lecture. *Monday 6 February 2023*. Absoluteness, upwards absoluteness, downwards absoluteness. Substructures in model theory. The language of set theory and its lack of function and constant symbols. Defined functions and constant symbols. Non-absoluteness of emptiness for arbitrary models of set theory. Transitive models of set theory. Formula classes: quantifier free, Δ_0, Σ_1, Π_1. Semantic formula classes: up to equivalence in a theory T. Formulas in Δ_0^T are absolute between transitive models of T. Lecture Notes.

Sixth Lecture. *Wednesday 8 February 2023*. Formulas in Σ_1^T are upwards absolute between transitive models of T. Formulas in Π_1^T are downwards absolute between transitive models of T. Set theoretic concepts that are in Δ_0: function, injection, bijection, cofinal subset; ordinal is in Δ_0^{ZFC}; set theoretic concepts that are in Π_0: cardinal, regular cardinal, inaccessible cardinal. Transitive models and inner models. Filters, ultrafilters, λ-completeness, principality. Existence of non-principal ultrafilters via Zorn's Lemma (no proof). Measurable cardinals. Measurable cardinals are inaccessible. Lecture Notes.

Seventh Lecture. *Monday 13 February 2023*. Infinitary languages: infinitary conjunctions and disjunctions, infinitary quantifiers, syntax and semantics. The expressive power of infinitary languages transcends first-order logic. Weakly compact cardinals. Weakly compact cardinals are inaccessible. The linearity phenomenon in the consistency strength hierarchy. Products and reduced products. Lecture Notes.

Eighth Lecture. *Wednesday 15 February 2023*. Infinitary languages: lengths of $\mathcal{L}_{\kappa\kappa}$-formulas; size of L_ξ. Łoś's theorem for infinitary languages. Every measurable cardinal is weakly compact. Keisler's Extension Property (KEP). Every weakly compact cardinal has the KEP (without proof). The smallest inaccessible cardinal is not the smallest weakly compact cardinal. Lecture Notes.

LC #1

Cofinality: regular, singular

Aleph fixed pts

Inaccessible cardinals

LC #2

ZERMELO'S Theorem

LC #3

Consistency strength hierarchy

Worldly cardinal

LC #4

Model theory in set theory: TVT, TCL.

LC #5

Absoluteness

LC #6

Sufficient conditions for fmlas to be absolute

Filters, measurable cardinals

$\boxed{\mathrm{MC} \longrightarrow \mathrm{IC}}$.

LC #7

Infinitary languages. WC

$\boxed{\mathrm{WC} \longrightarrow \mathrm{IC}}$

LC #8 Łoś for $\mathcal{L}_{\kappa\kappa}$.

$\boxed{\mathrm{MC} \longrightarrow \mathrm{WC}}$

Key results:

- Zermelo's Thm } less than least inaccessible
- Least worldly } less than least inaccessible
- $\mathrm{MC} \longrightarrow \mathrm{IC}$
- $\mathrm{WC} \longrightarrow \mathrm{IC}$ / $\mathrm{MC} \longrightarrow \mathrm{WC}$.

FROM MOODLE

Ninth Lecture. *Monday 20 February 2023.* Every weakly compact cardinal has the Keisler extension property. Reflection and bootstrapping of reflection. Below every weakly compact cardinal there are unboundedly many inaccessible cardinals. Erdős arrow notation. Relation to Ramsey's theorem. Finite partition cardinals. A cardinal is weakly compact if and only if it is finite partition (without proof). Finite partition cardinals are inaccessible. Lecture Notes.

Tenth Lecture. *Wednesday 22 February 2023.* Closure under diagonal intersections. Normal filters. Measurable cardinals carry a normal ultrafilter (no proof yet). Measurable cardinals with a normal ultrafilter are finite partition. Measurable cardinals and elementary embeddings: $\kappa < \lambda$ with κ measurable and λ inaccessible is strictly stronger than MC. Ultrapower of $\mathbf{V}_\lambda$ with U; basic properties; the ultrapower embedding; wellfoundedness of the ultrapower; transitive ultrapower M as Mostowski collapse of the ultrapower; $M \subseteq \mathbf{V}_\lambda$. Lecture Notes (note: correction of proof on page 4).

Eleventh Lecture. *Monday 27 February 2023.* The ordinals of M are λ. The embedding is the identity on $\mathbf{V}_\kappa$. The embedding is not the identity: $j(\kappa) > \kappa$. Critical point of an embedding. $\mathbf{V}_{\kappa+1} \subseteq M$. If λ is the least inaccessible above κ, then $\mathbf{V}_\lambda \neq M$. In $\mathbf{V}_\lambda$, the cardinality of $j(\kappa)$ is at most 2^κ, so $j(\kappa)$ is not measurable. Since $U \notin M$, we have that $\mathbf{V}_{\kappa+2} \not\subseteq M$ (no proof, cf. Example Sheet #3). Reflection arguments: new proof that there are unboundedly many inaccessibles below a measurable. Lecture Notes.

Twelfth Lecture. *Wednesday 1 March 2023.* Is κ measurable in M? First approach: meta-argument that "κ is measurable in M" cannot be provably. Second approach: surviving cardinals; reflection argument to show that a surviving cardinal has unboundedly many measurable cardinals below. Fundamental Theorem on Measurable Cardinals. Construction of ultrafilter U_j. Proof that U_j is κ-complete non-principal ultrafilter on κ. This filter is always normal (no proof yet). Lecture Notes.

Thirteenth Lecture. *Monday 6 March 2023.* The filter derived from an elementary embedding is always normal. Every measurable cardinal carries a normal ultrafilter. Strengthening of the reflection arguments: if U is a normal ultrafilter on κ, then the set of inaccessible cardinals below κ is in U. Getting rid of the inaccessible cardinal: Scott's trick. Constructing the ultrapower as a transitive class of sets. Definability of the ultrapower and the embedding from U. Elementarity and the nondefinability of truth. Elementarity as a schema of axioms. The Fundamental Theorem on Measurable Cardinals as a theorem schema. Class theories: von Neumann-Bernays-Gödel (NGB) and Kelley-Morse (KM) without definitions. Lecture Notes.

Fourteenth Lecture. *Wednesday 8 March 2023.* The least measurable cardinal is not the least weakly compact: proof that κ is weakly compact in the ultrapower M. Reflection arguments show that the property of being weakly compact reflects below a measurable. Stable properties: inaccessibility is 1-stable and measurability is 2-stable. 1-stable properties always reflect below a measurable, 2-stable properties not necessarily. Witness formulas and witness objects: measurability of κ has a witness object in $\mathbf{V}_{\kappa+2}$. Strength of embeddings: every embedding is 1-strong. Strong cardinals. Second example of the fact that the maps $U \mapsto j_U$ and $j \mapsto U_j$ are not inverses of each other. Any α-stable property reflects below an α-strong cardinal. Being an α-strong cardinal cannot have a witness object in $\mathbf{V}_{\kappa+\alpha}$. Lecture Notes.

Fifteenth Lecture. *Monday 13 March 2023.* The survival relation on normal ultrafilters. Mitchell characterisation lemma. The survival relation is a well-founded, irreflexive, transitive relation (without proof). Mitchell order of ultrafilters. Cardinals with higher Mitchell order and their witness objects. Being of positive Mitchell order reflects at a 2-strong cardinal. Strong cardinals. Reinhard cardinals. Kunen's Inconsistency. The Erdős-Hajnal theorem on ω-Jónsson functions. Proof of Kunen's Inconsistency. Lecture Notes.

Sixteenth Lecture. *Wednesday 15 March 2023.* Kunen's lemma implies that there cannot be an elementary embedding $j : \mathbf{V}_{\lambda+2} \to \mathbf{V}_{\lambda+2}$. Axiom candidates close to the Kunen inconsistency: I1 and I3. Algebraic nature of these axioms and connection to braid groups. Supercompactness: the supercompact analogue of Reinhardt cardinals cannot exist either. A reflection-based ordering of strength of cardinal properties and its problems: identity crises. The large cardinal hierarchy: an overview. Lecture Notes.

LC#9

$\boxed{\text{WC} \longrightarrow \text{KEP}}$

REFLECTION

$\boxed{\text{FPC} \longrightarrow \text{IC}}$

LC#10

Normal filter.

$\boxed{\text{MC} \longrightarrow \text{FPC}}$

(modulo normality)

ULTRAPOWER OF V_λ

LC#11

$V_{\kappa+1} \subseteq M$

$V_{\kappa+2} \not\subseteq M$

LC#12

Surviving cardinals

$\boxed{\text{FTMC}}$

LC#13

MC $\longrightarrow$ there is a normal ultrafilter

Scott's Trick & Class Theories

LC#14

WC $\not\longrightarrow$ MC

Stability

Strength.

Key results (continued):

- WC $\longrightarrow$ KEP
- FPC $\longrightarrow$ IC
- MC $\longrightarrow$ FPC
- FTMC
- Kunen's Inconsistency

LC#15

Mitchell order

Mitchell characterisation lemma

Reinhardt Cardinals

LC#16

$\boxed{\text{Kunen's Inconsistency}}$

Supercompactness

$$\mathrm{ZFC} <_{\mathrm{Cons}} \mathrm{ZFC} + \mathrm{IC}$$

is false if ZFC is inconsistent.

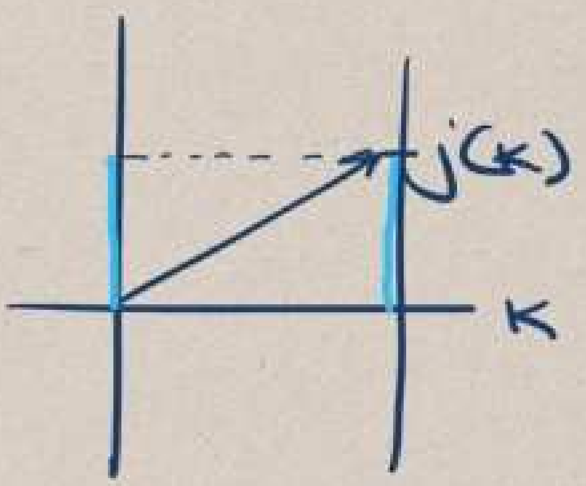
$$T_0 < T_1 < T_2 < T_3$$

$$j : V_\lambda \longrightarrow M$$

$\mathrm{crit}(j) = \kappa$

for all $\alpha < \kappa$ $\quad j(\alpha) = \alpha$

$\qquad j(\kappa) > \kappa$



For each $\kappa < \alpha < j(\kappa)$, we know that there is no γ s.t.

$$j(\gamma) = \alpha.$$

They must come from some $f : \kappa \longrightarrow V_\lambda$ s.t.

$$\alpha = (f).$$

Since $\alpha < j(\kappa) = (c_\kappa)$

So $\{\gamma ; f(\gamma) < \kappa\} \in U$, so w.l.o.g. assume $f : \kappa \longrightarrow \kappa$.

Not constant on a set in U.

Suppose U is normal.

Then $(\mathrm{id}) = \kappa$, so then

e.g., $f(\alpha) := \alpha + \alpha$ creates

$$(f) = \kappa + \kappa = (\mathrm{id}) + (\mathrm{id}).$$

$$\begin{aligned} f : \kappa &\longrightarrow \kappa \\ \alpha &\longmapsto \alpha + 1 \end{aligned}$$

$$(f) = \kappa + 1.$$

The language of set theory has only one nonlogical symbol $\in$. Everything else is defined in terms of it.

In particular: x is the power set of y

$$\underline{\Phi}(x,y) :\Longleftrightarrow \forall z \left(z \in x \longleftrightarrow \forall w \left(w \in z \longrightarrow w \in y \right) \right)$$

$M \models \kappa$ is a strong limit cardinal

$\forall \lambda (\lambda < \kappa \longrightarrow$ there is no surjection from λ to $\kappa)$

$\wedge\ \forall \lambda \forall p (\lambda < \kappa \wedge \underline{\Phi}(p, \lambda) \longrightarrow$ there is no surjection from p to $\kappa)$

If λ is a limit ordinal, consider

$$\boxed{V_\lambda \subseteq V.}$$

If $x, y \in V_\lambda$ are any two sets, then if $f : x \longrightarrow y$ in V, then $f \in V_\lambda$.

Let $\alpha := \rho(x), \beta := \rho(y) < \lambda$ (both $< \lambda$)

then $\gamma := \max(\alpha, \beta) < \lambda$.

Therefore $f \in \mathcal{P}(\mathcal{P}(V_\gamma)) = V_{\gamma+2} \subseteq V_\lambda$.

<u>Remember</u> $V_{\omega+\omega} \models Z$ + negation of Replacement

Note that $\mathcal{P}(\omega) \in V_{\omega+2} \subseteq V_{\omega+\omega}$.

However $\mathrm{Ord} \cap V_{\omega+\omega} = \omega+\omega$, so in $V_{\omega+\omega}$, there is no ordinal in bijection with $\mathcal{P}(\omega)$.

Standard notation

Ordinals

$\alpha \longmapsto \alpha+1 = \alpha \cup \{\alpha\}$

$\alpha,\beta \longmapsto \alpha+\beta$

$\alpha,\beta \longmapsto \alpha\cdot\beta$

$\alpha,\beta \longmapsto \alpha^{\beta}$

Cardinals

$\kappa \longmapsto \kappa^{+} =$ Hartogs aleph of κ

$\kappa,\lambda \longmapsto \kappa+\lambda$

$\kappa,\lambda \longmapsto \kappa\cdot\lambda$

$\kappa,\lambda \longmapsto \kappa^{\lambda}$

Same notation, but (very) different functions.

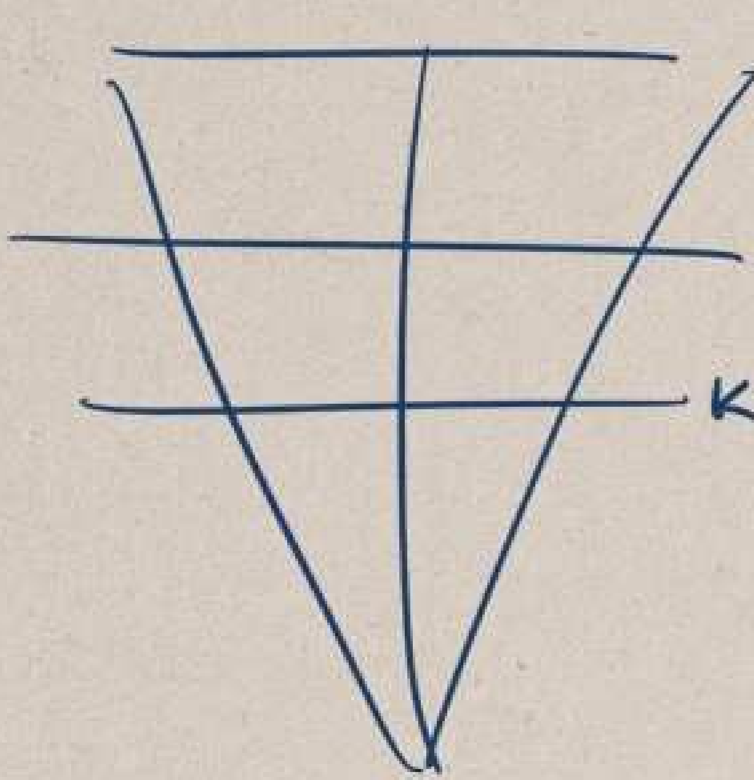
If α,β ordinals

$$|\alpha+\beta| = |\alpha|+|\beta|$$

$$|\alpha\cdot\beta| = |\alpha|\cdot|\beta|$$

$$|2^{\omega}| = \aleph_0 \neq 2^{\aleph_0}$$

ordinals



λ

$\kappa_1 \rightsquigarrow M_1$

$\kappa_0 \rightsquigarrow M_0$

$j_0(\kappa_0) > \kappa_0$

$j_1(\kappa_1) > \kappa_1$

$\boxed{j_1(\kappa_0)}$

EASY.

$crit(j_0) = \kappa_0$

$crit(j_1) = \kappa_1$

$j_0(\kappa_1)$

If $\boxed{\alpha < j_0(\kappa_1)} = (C_{\kappa_1})$.

If $(f) = \alpha$, then $f : \kappa_0 \longrightarrow \underset{\text{regular}}{\kappa_1}$ [w.l.o.g.]

There is some $\underset{< \kappa_1.}{\eta}$ s.t. $f : \kappa_0 \longrightarrow \eta$

So $|j_0(\kappa_1)| \leq \left| \bigcup_{\eta < \kappa_1} \underbrace{\{f ; \kappa_0 \longrightarrow \eta\}}_{|\kappa_0^\eta| < \kappa_1} \right| \leq \kappa_1$.