LARGE CARDINALS

XVI

LECTIO
ULTIMA
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LOOSE ENDS

1. Large cardinals & GCH
2. Witness objects
3. The annoying inaccessible above

1. Large cardinals & GCH

Theorem (Scott) If κ is measurable and GCH holds below κ ($\forall \mu < \kappa \;\; 2^\mu = \mu^+$) then $\kappa^+ = 2^\kappa$.

Proof. If GCH holds below κ, then by elementarity,

$$M \models \text{GCH holds below } j(\kappa)$$
$$M \models 2^\kappa = \kappa^+.$$

By our analysis, we know $V_{\kappa+1} \subseteq M$, so $\wp(\kappa) \subseteq M$ and thus $\kappa^+ = (\kappa^+)^M = (2^\kappa)^M = 2^\kappa$.

q.e.d.

We'll improve this for supercompact cardinals:

Prop. If κ is γ-supercompact and μ is such that $\mu \leq \gamma$ and GCH holds below κ, then $2^{\mu} = \mu^{+}$.

Proof. Essentially the same proof if $\gamma < j(\kappa)$.
By our proof of Kunen's inconsistency, we can w.l.o.g. assume that $\gamma < j(\kappa)$: we saw that for any j, j is not $\hat{\kappa}$-supercompact, so we know that there is some n s.t. $\gamma < \kappa_n$ (nth iterate of j on κ)
So we can work with the nth iterate of j instead which move κ beyond γ.

Then we give Scott's argument:
Because GCH holds below κ,
$M \models$ GCH holds below $j(\kappa)$
$\Longrightarrow M \models \mu^{+} = 2^{\mu}$.
Use γ-supercompactness to see that $\mathcal{P}(\mu)^{M} = \mathcal{P}(\mu)$.
q.e.d.

② Witness objects

Fundamental theorem : TFAE

(i) $\exists \underbrace{j : V_\lambda \longrightarrow M}_{\in V_{\lambda+1}}$ elementary $\mathrm{crit}(j) = \kappa$

(ii) $\exists \underbrace{U}_{\in V_{\kappa+2}}$ κ-complete nontrivial on κ uf

Observation The power of the fundamental theorem is that the elementarity is witnessed by an object in $V_{\kappa+2}$.

This is used in REFLECTION ARGUMENTS

E.g. if κ is 2-strong, then κ is the κ-th measurable

This uses the fact that measurability is witnessed by something in $V_{\kappa+2}$ which then is preserved by the (2-strong) embedding.

? Witness objects for
- supercompactness
- strength
- strong compactness

WITNESS OBJECTS FOR SUPERCOMPACTNESS

KANAMORI THE HIGHER INFINITE

22.7 Theorem (Solovay, Reinhardt). *If $\kappa \leq \gamma$, κ is γ-supercompact iff there is a normal ultrafilter over $\mathcal{P}_\kappa\gamma$.* ⊣

One observation that can now be made is that supercompactness does not entail the existence of greater large cardinals:

22.8 Exercise.

(a) If κ is supercompact and $\lambda > \kappa$ is inaccessible, then $V_\lambda \models \kappa$ is supercompact.

(b) If Con(ZFC + $\exists\kappa$(κ is supercompact)), *then*
Con(ZFC + $\exists\kappa$(κ is supercompact $\wedge$ $\neg\exists\lambda$($\lambda > \kappa$ $\wedge$ λ is inaccessible))). ⊣

$$\mathcal{P}_\kappa\gamma := \{X \in \mathcal{P}(\gamma) ; |X| \leq \kappa\}$$

Where does an ultrafilter on $\mathcal{P}_\kappa\gamma$ live?

In $V_{\gamma+2}$.

So Theorem 22.7 is like the Fundamental Theorem: it reduces a statement about elementarity to a statement about objects in $V_{\gamma+2}$.

This allows us to show (Exercise 22.8) that the assumption

$\exists\kappa\exists\lambda$ $\kappa<\lambda$ and κ has property Φ and λ is strongly inaccessible

is STRICTLY STRONGER than ΦC.

WITNESS OBJECTS FOR STRONGLY COMPACT CARDINALS.

Def. A cardinal κ is called γ-compact if there is a (so called) fine ultrafilter on $P_\kappa \gamma$.

KANAMORI:

22.17 Theorem. *If $\kappa \le \gamma$, the following are equivalent:*

(a) κ is γ-compact.

(b) There is a $j\colon V \prec M$ with $\mathrm{crit}(j) = \kappa$ such that: for any $X \subseteq M$ with $|X| \le \gamma$, there is a $Y \in M$ such that $Y \supseteq X$ and $M \models |Y| < j(\kappa)$.

(c) For any set S, every κ-complete filter over S generated by at most $|\gamma|$ sets can be extended to a κ-complete ultrafilter over S.

Reverse direction of the Keisler-Tarski theorem:

Let κ be a cardinal. If for any set S, every κ-complete filter over S can be extended to a κ-complete ultrafilter over S, then κ is strongly compact.

Proof. Kanamori, *The Higher Infinite*, p. 37 (Proposition 4.1):

Suppose now that $\Sigma = \{\sigma_\alpha \mid \alpha < \lambda\}$ is a κ-satisfiable collection of $L_{\kappa\kappa}$ sentences. Recall that $\mathcal{P}_\kappa\lambda = \{x \subseteq \lambda \mid |x| < \kappa\}$. For any $x \in \mathcal{P}_\kappa\lambda$, let $\mathcal{M}_x$ be a structure for the language of Σ so that $\mathcal{M}_x \models \bigwedge_{\alpha\in x}\sigma_\alpha$. With the availability of $\mathcal{M}_\lambda$ we can assume that $\lambda \ge \kappa$. As

$$\{\{x \in \mathcal{P}_\kappa\lambda \mid y \subseteq x\} \mid y \in \mathcal{P}_\kappa\lambda\}$$

generates a κ-complete filter over $\mathcal{P}_\kappa\lambda$ by the regularity of κ, let U be a κ-complete ultrafilter over $\mathcal{P}_\kappa\lambda$ extending this filter. Consider the ultraproduct $\mathcal{M} = \prod_{\mathcal{P}_\kappa\lambda}\mathcal{M}_x/U$. It is straightforward to check that, essentially by the same proof as for $L_{\omega\omega}$, Łoś's Theorem 0.6 holds for $L_{\kappa\kappa}$ and ultraproducts by κ-complete ultrafilters. Since for any $\alpha < \lambda$, Witness objects for strong cardinals:

EXTENDERS

A (κ,γ)-extender is a family of ultrafilters

$$\{E_a \,;\, a \subseteq \gamma \text{ finite}\}$$

with certain coherence properties.

These extenders can be used as witness objects for strong embeddings.

$\longrightarrow$ § 26 Kanamori.

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③ The ~~annoying~~ inaccessible.

FUNDAMENTAL THEOREM TFAE

(i) $\boxed{\exists j : V_\lambda \longrightarrow M \text{ elem. } \mathrm{crit}(j)=\kappa}$

(ii) $\exists U$ κ-complete nontrivial uf. on κ

In order to remove the inaccessible, we'd like to be able to express in a model of set theory that there is an elementary embedding.

If we are working in a fixed model
$V \models ZFC$
then both j and M would be proper classes and therefore we cannot quantify over ~~th~~ them.

If V is a model of set theory that is a "set" in our meta-set theory, then its classes are just $\mathcal{P}(V)$ ← meta power set.

Many classes that matter in practice are definable: if φ is a formula with $n+1$ free variables and $\vec{x} \in V^n$, then we can apply meta-separation to get

$$C_{\varphi, \vec{x}} := \{x \in V ; V \models \varphi(x, \vec{x})\}$$

There are $|V|$ definable classes over V, but $2^{|V|}$ classes over V.

Reformulation of the FUNDAMENTAL THM in terms of definable classes as a meta-theorem:

TFAE

(i) there is φ and $\vec{x}$ s.t. $C_{\varphi,\vec{x}}$ is an elem. embedding with crit. pt. κ

(ii) $\exists\, U$ κ-complete nontrivial over κ

Infinitely many formulas.

Alternatively:
This could be done in a first-order way in some standard class theory
(NBG: von Neumann–Bernays–Gödel)

Now to the proof of this new version of the FUNDAMENTAL THM:

(i) $\Longrightarrow$ (ii) Works exactly as before, replacing all references to j by appropriate terms involving φ and $\vec{x}$.

(ii) $\Longrightarrow$ (i) $\qquad\qquad \Phi$

We'd like a formula using U as a parameter s.t.

$$x \in M \iff \Phi(x)$$

$\iff$ x is the image of the Mostowski collapse of some $[f]_U$.

where $f : \kappa \longrightarrow V$.

[Note that by replacement, $f \in V$.]

However, $[f]_U$ is not an element of V: if f is any function and $x \in V$ is any set then

$$f_x : \alpha \longmapsto \begin{cases} x & \text{if } \alpha = 0 \\ f(\alpha) & \text{if } \alpha \neq 0 \end{cases}$$

with $f_x \sim_U f$, but the ranks of the f_x are unbounded in V.

Solution

Scott's Trick

Consider the class

$$[f]_U \subseteq V$$

Find α minimal s.t. $\underbrace{[f]_U \cap V_\alpha}_{\text{scott}(f)} \neq \emptyset$

Clearly if $f \not\sim_U g$, then

$$\text{scott}(f) \cap \text{scott}(g) = \emptyset.$$

$\longrightarrow$ Our formula Φ is:

$\Phi(x) \iff$ x is the Mostowski collapse of some $\text{scott}(f)$ for some $f : \kappa \longrightarrow V$.

REVISION EXAMPLE CLASS

20 MAY 2022 3:30-5:30pm MR9