LARGE CARDINALS

PENULTIMATE LECTURE

XV

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INTERNATIONAL DAY OF MATHEMATICS
MARCH 14

STRENGTHENINGS OF MEASURABILITY

(1) LIMIT PROCESSES
α-measurability

(2) MITCHELL ORDER
$o(\kappa) \geq u$
(a) If $o(\kappa) \geq 1$, then κ is κ-measurable.
(b) $o(\kappa) \geq 0 \Longleftrightarrow \kappa$ is measurable

(3) STRENGTH
j is α-strong $:\Longleftrightarrow \kappa = \mathrm{crit}(j)$ & $V_{\kappa+\alpha} \subseteq M$
κ is α-strong $:\Longleftrightarrow$ there is a j with $\mathrm{crit}(j) = \kappa$ that is α-st.

κ is 1-strong $\Longleftrightarrow$ κ is measurable

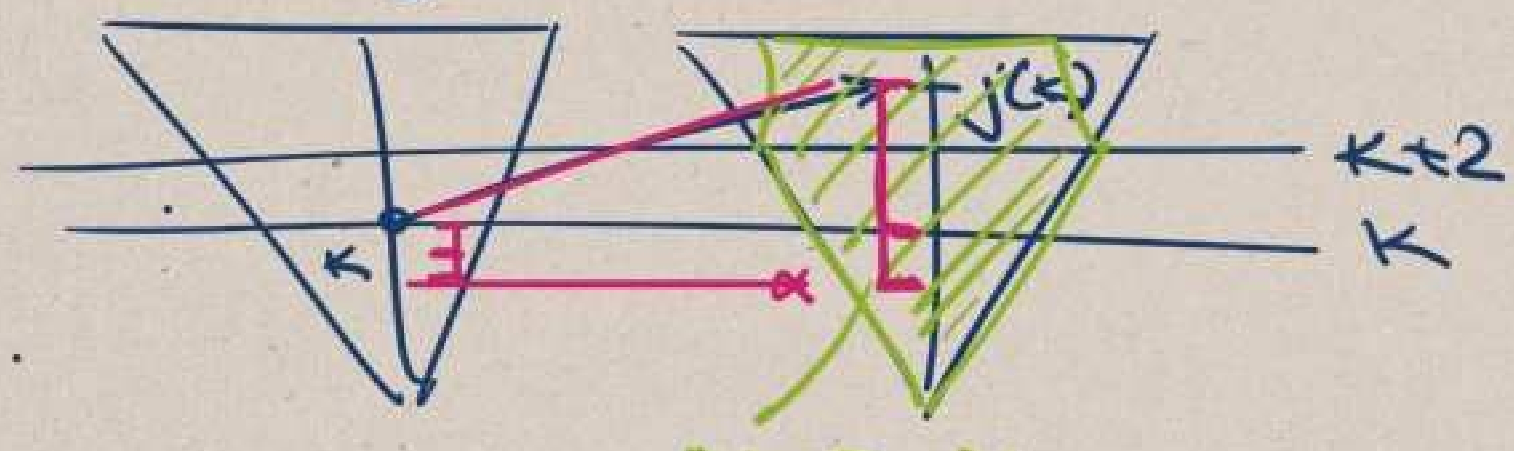
(4) · SUPERCOMPACTNESS

In particular, if κ is measurable with ultrapower embedding j, then

j is 1-strong $\quad V_{\kappa+1} \subseteq M$

but not 2-strong $\quad V_{\kappa+2} \not\subseteq M$.

Again by our standard technique (reflection), 2-strength is (much) stronger than measurability:



$M \supseteq V_{\kappa+2}$

κ is measurable (by the reverse direction of the Fundamental Thm), thus there is a κ-complete nontrivial uf. U on κ.

But $U \in V_{\kappa+2} \subseteq M$.

So $M \models \kappa$ is measurable.

Thus $\exists \mu \ \alpha < \mu < j(\kappa) \ \mu$ is measurable

$\overset{\text{elem.}}{\Longrightarrow} V_\lambda \models \exists \mu \ \alpha < \mu < \kappa \ \mu$ is measurable.

Theorem (without proof)

If κ is 2-strong then for each $n \in \mathbb{N}$, $o(\kappa) \geq n$.

On ES#3, (44), you'll see that if κ is 2-strong and $o(\kappa) \geq n$, then κ is not the least cardinal with $o(\kappa) \geq n$.

Definition A cardinal κ is called strong if it is α-strong for all ordinals α.

IMPORTANT REMARK If κ is strong, then: For each α, there is $j_\alpha : V_\lambda \longrightarrow M_\alpha$ s.t. $V_{\kappa+\alpha} \subseteq M_\alpha$.

It does not mean that this is witnessed by a single embedding!

Def. A cardinal κ is called a Reinhardt cardinal if there is an embedding j with $\mathrm{crit}(j) = \kappa$ that is α-strong for all α. [I.e., $M = V_\lambda$.]

Theorem (Kunen 1971)

There are no Reinhardt cardinals.

KUNEN'S INCONSISTENCY.

Let's make the statement more concrete:

Theorem Suppose $j: V \longrightarrow M$ is elementary with $\mathrm{crit}(j) = \kappa$. Define

$$\kappa_0 := \kappa$$
$$\kappa_{i+1} := j(\kappa_i)$$
$$\hat{\kappa} := \bigcup_{i \in \mathbb{N}} \kappa_i .$$

In particular, $\hat{\kappa}$ is the least fixed pt of $j > \kappa$.

Then there is $X \in V_{\hat{\kappa}+1}$ s.t. $X \notin M$.

In particular, j is not $\hat{\kappa}+1$-strong.

Proof. We'll need a combinatorial lemma which we won't prove.

If X is a totally ordered set, we write $[X]^\omega$ for the set of all strictly increasing functions from ω into X.

Let μ be any cardinal and

$$f : [\mu]^\omega \longrightarrow \mu$$

Then f is called <u>ω-Jónsson</u> if for all $X \subseteq \mu$ s.t. $|X| = \mu$ we have

$$\{ f(y) ; y \in [X]^\omega \} = \mu .$$

<u>Combinatorial LEMMA</u> (Erdős-Hajnal 1966)

Every infinite cardinal has an ω-Jónsson function.

[The proof is in Kanamori's book & not too complicated.]

NOW THE PROOF THAT THERE IS $X \notin M$ s.t. $X \subseteq V_{\hat{\kappa}}$.

Fix $j : V_\lambda \longrightarrow M$

By Erdős-Hajnal fix an ω-Jónsson function

$$f : [\hat{\kappa}]^\omega \longrightarrow \hat{\kappa} .$$

By elementarity,

$M \models j(f)$ is an ω-Jónsson fn on $j(\hat{\kappa}) = \hat{\kappa}$.

Let $\underline{X := \{j(\alpha)\,;\ \alpha \in \kappa\}} \subseteq V_{\hat\kappa}$.

[because $\hat\kappa$ is a j-fixed pt].

Clearly, $|X| = \kappa$. Let's assume towards a contradiction that $X \in M$.

By definition, this means

$$\{\boxed{j(f)(y)}\,;\ y \in [X]^\omega\} \boxed{= \hat\kappa.}$$

$$= \{j(f(x))\,;\ x \in [\kappa]^\omega\}$$

$$\boxed{= \{j(\alpha)\ ;\ \alpha \in \hat\kappa\}}$$ [by the fact that f is ω-Jónsson]

If $y \in [X]^\omega$, then $y = \{j(\alpha_n)\,;\ n \in \mathbb{N}\}$ where $\alpha_n \in \kappa$.

But then $x := (n \longmapsto \alpha_n)$ is a function from ω into $\hat\kappa$

By definition, we have that $j(x) = y$.

Thus $j(f)(y) = j(f)(j(x)) \overset{\text{elem.}}{=} j(f(x))$.

So: $\hat\kappa = \{j(\alpha)\,;\ \alpha \in \hat\kappa\}$.

But that's false since $\kappa \in \hat\kappa$, but there is no $\alpha \in \hat\kappa$ s.t. $\kappa = j(\alpha)$. Contradiction! q.ed

(4) Supercompactness

If μ is a cardinal and M an inner model, we say that M is closed under μ-sequences if $M^\mu \subseteq M$.

We say an elementary embedding

$$j : V_\lambda \longrightarrow M$$

is μ-supercompact if M is closed under μ-sequences.

Lemma If an embedding j is 2^κ-supercompact where $\mathrm{crit}(j) = \kappa$, then it is 2-strong.

Proof. If $X \in V_{\kappa+2}$, $X \subseteq V_{\kappa+1}$,

$$|X| \leq |V_{\kappa+1}| = 2^{|V_\kappa|} = 2^\kappa.$$

So since $V_{\kappa+1} \subseteq M$, we have that X is a 2^κ-sequence. q.e.d.

We'll that the ultrapower embedding of a measurable cardinal is closed under κ-sequences.

<u>Theorem</u> If κ is measurable and M the ultrapower, then $M^\kappa \subseteq M$.

<u>Proof.</u> If $x \in M^\kappa$, then

$$x : \kappa \longrightarrow M$$

$$x = ((f_\alpha) ; \alpha < \kappa)$$

f_α representative of $x(\alpha)$.

Let $h : \kappa \longrightarrow \kappa$ be s.t. $(h) = \kappa$.

Define a function g with $\mathrm{dom}(g) = \kappa$ that yields the right objects:

$g(\xi)$ is the function g_ξ s.t.

$$\mathrm{dom}(g_\xi) = h(\xi)$$

and for all $\alpha \in \mathrm{dom}(g_\xi)$

$$g_\xi(\alpha) = f_\alpha(\xi).$$

by Łoś

We know in M that:

① (g) is function

② $\mathrm{dom}((g)) = (h) = \kappa$

③ if $\alpha < \kappa$ $(g)(\alpha) = (f_\alpha)$

So $(g) = x$, and therefore $x \in M$. q.e.d.

Def. • A cardinal κ is called μ-supercompact if there is an embedding
$$j : V_\lambda \longrightarrow M \quad \text{s.t.}$$
j is μ-supercompact.

• A cardinal κ is called supercompact if it is μ-supercompact for all μ.

Observe (1) measurable $\Longleftrightarrow$ κ-supercompact

(2) The ultrapower embedding is not κ^+-supercompact [ES#3, (45)]

(3) Kunen's inconsistency shows that no embedding with $\mathrm{crit}(j) = \kappa$ is $\hat{\kappa}$-supercompact.