Lecture XIV

Large Cardinals

9 March 2022

L an $\mathcal{L}_{\kappa\kappa}$-language with set $\underline{S = \{s_\alpha ; \alpha < \kappa\}}$ of non-logical symbols

$\Phi \subseteq L$ $\quad |\Phi| \leq \kappa$ $\quad$ [From the proof: measurable $\Rightarrow$ w.c.]

$M \models \Phi$ is κ-satisfiable

$\Downarrow$

$V_\lambda \models \Phi$ is satisfiable

Let $\boxed{N \models \Phi}$ with $\underline{N \in V_\lambda}$.

<u>ARGUED</u> if $|N| \leq \kappa$, then $N \in M$, and so $M \models \Phi$ is satisfiable.

$\boxed{\underline{\text{MISSING}} \quad |N| \leq \kappa.}$

A Löwenheim-Skolem theorem for $\mathcal{L}_{\kappa\kappa}$

$\longrightarrow$ Example Sheet #3.

We're going to prove the fact that

$$M \models \Phi \text{ is satisfiable}$$

differently.

<u>Idea</u>: Use $j(N)$ and show that

$$M \models \text{“} j(N) \models \Phi \text{”}.$$

Main idea is correct, but one needs to be careful.

$$S = \{s_\alpha ; \alpha < \kappa\}$$

Because $|L| \leq \kappa$, we can write

$$\Phi = \{\varphi_\alpha ; \alpha < \kappa\}.$$

Let's assume that both $S, \Phi \subseteq V_\kappa$.

$V_\lambda \models N$ is an L-structure and for all $\varphi \in \Phi$, "$N \models \varphi$" ①

$M \models j(N)$ is a $\boxed{j(L)}$-structure and for all $\varphi \in \boxed{j(\Phi)}$ ②, "$j(N) \models \varphi$".

<u>Problems are</u>: ① $j(L)$ might not be L.

It is not, since L was an $\mathcal{L}_{\kappa\kappa}$-language, so $j(L)$ is an $\mathcal{L}_{j(\kappa)j(\kappa)}$-language.

② $j(\Phi)$ might not be Φ.

(1) What is $j(\mathcal{L})$?
It is an $\mathcal{L}_{j(\kappa)j(\kappa)}$-language with non-logical symbols $j(S)$.

$\alpha<\kappa$

$V_\lambda \models$ the αth symbol in S is s_α

$\Longrightarrow M \models$ the $\underset{\alpha}{\underset{\parallel}{j(\alpha)}}$th symbol in $j(S)$ is $\underset{s_\alpha}{\underset{\parallel}{j(s_\alpha)}}$

[since $s_\alpha \in V_\kappa$ & $j \restriction V_\kappa = id$]

So $S \subseteq j(S)$.

Therefore we have an S-reduct of the $j(S)$-structure $j(N)$. Call it $\overline{N}$.

(2) What is $j(\Phi)$?

$V_\lambda \models$ the αth formula in Φ is φ_α

$\Downarrow$

$M \models$ the $\underset{\alpha}{\underset{\parallel}{j(\alpha)}}$th formula in $j(\Phi)$ is $\underset{\varphi_\alpha}{\underset{\parallel}{j(\varphi_\alpha)}}$

So $\Phi \subseteq j(\Phi)$.

[since $\Phi \subseteq V_\kappa$, $j \restriction V_\kappa = id$]

Together:

$$M \models \text{"}j(N) \models \Phi \subseteq j(\Phi)\text{"}$$

So $M \models \Phi$ is satisfiable.

q.e.d.

STRENGTHENINGS OF MEASURABILITY

Four possible directions:

1. Limit processes.
2. Survival
3. Strength
4. Supercompactness

1 Limit processes

Definition κ is called 0-measurable if it is measurable

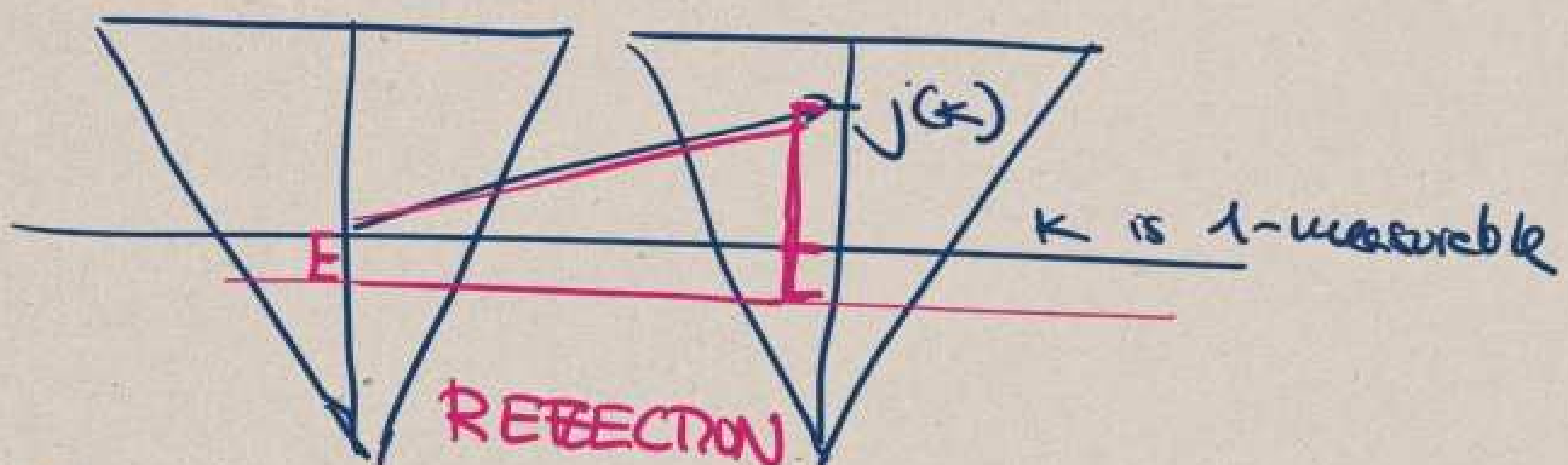
κ is called $\alpha+1$-measurable if it's α-measurable and $\{\mu<\kappa;\ \mu \text{ is } \alpha\text{-measurable}\}$ is unbounded in κ

κ is λ-measurable (λ limit) if κ is α-meas f.a. $\alpha<\lambda$.

We saw last time (Lecture $\underline{\overline{XIII}}$):

If κ is surviving (i.e., $M \models \kappa$ is measurable) then $\{\alpha < \kappa ; \alpha$ is measurable$\}$ is unbounded in κ.

$\Longrightarrow$ κ is 1-measurable.



So:

$M \models \exists \beta\,(\alpha < \beta < j(\kappa)$ and β is 1-measurable$)$

$\Longrightarrow$ $V_\kappa \models \exists \beta\,(\alpha < \beta < \kappa$ and β is 1-measurable$)$

So κ is 2-measurable.

An induction shows:

Surviving cardinals κ are κ-measurable.

② Survival

Survival is really a relation on ultrafilters on κ:

NOTE
It is not obvious that $<$ is transitive!:

$U < V$ "V survives U"

$\iff V \in M_U = \pi[\mathrm{Ult}(V_\lambda, U)]$.

So whether κ is surviving can be expressed by saying that there are two ultrafilters, one of which survives the other.

What does it mean that $U < V$?
This means that there is $g : \kappa \longrightarrow V_\lambda$ s.t. $(g)_U = V$.
Let $\underline{f : \kappa \longrightarrow \kappa}$ be such that $(f)_U = \kappa$.
Then

$M_U \models (g)_U$ is a $(f)_U$-complete uf. on $(f)_U$

$\overset{Łoś}{\iff} \{\alpha \,;\, g(\alpha)$ is an $f(\alpha)$-complete uf. on $f(\alpha)\} \in U$.

Note that $f(\alpha) < \kappa$, so $g(\alpha) \in V_\kappa$.

So both $f, g \in V_{\kappa+1}$.

Consequence If M is any inner model s.t. $V_{\kappa+1} \subseteq M$ and $U, V \in M$, then

$$M \models U < V$$

$$\iff V_\lambda \models U < V.$$

Definition A cardinal κ has Mitchell rank $\geq n$ if there are ultrafilters $U_0, U_1, \dots, U_n$ κ-complete nontrivial s.t.

NOTE Due to the transitivity issue, we need: f.a. $i<j$, $U_i < U_j$.

$$U_0 < U_1 < \dots < U_n$$

It has Mitchell rank $= n$ if it has Mitchell rank $\geq n$ and not $\geq n+1$.

[Remark. This definition can be easily extended into the transfinite.]

We write $o(\kappa) = n$ for κ has Mitchell rank n.

Clearly: Having Mitchell rank ≥ 0 is equivalent to being measurable.

Observe that

$o(\kappa) \geq 1$ is equivalent to our earlier notion of surviving.

Let's show that $o(\kappa) \geq n+1$ is strictly stronger than $o(\kappa) \geq n$:

[Let's do this for $n=1$, rest is induction.]

Suppose $o(\kappa) \geq 2$, so find

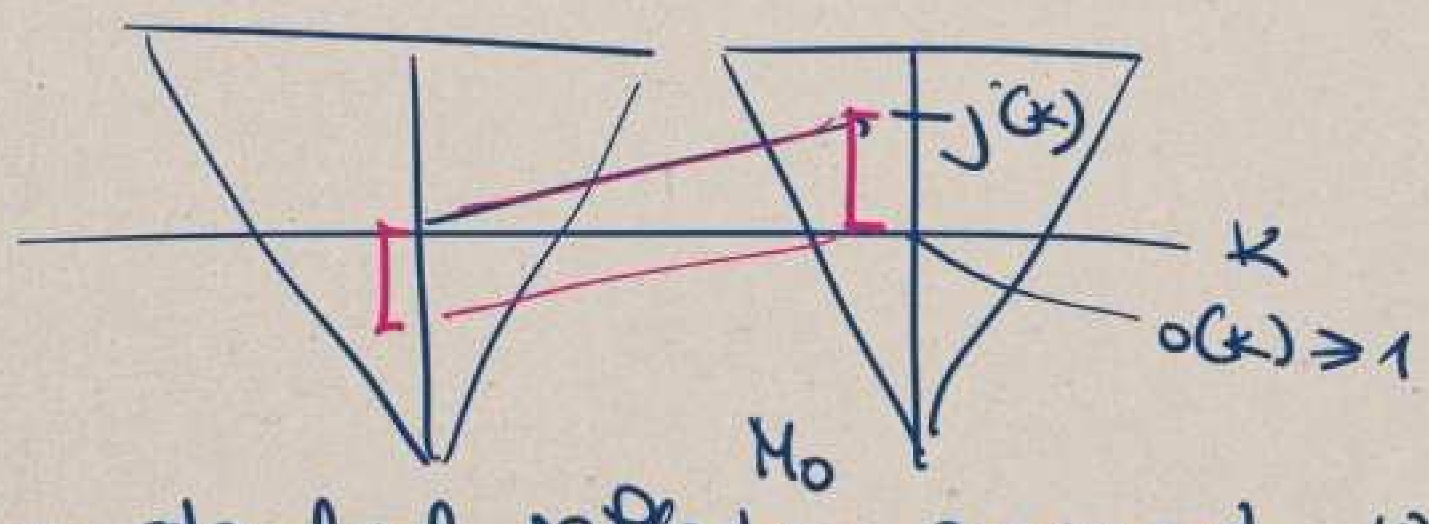
$$U_0 < U_1 < U_2$$

NOTE with $U_0 < U_2$!

Let M_0 be the ultrapower by U_0, then $V_{\kappa+1} \subseteq M_0$, so by the consequence on p. 7, we get

$$M_0 \models U_1 < U_2$$

$$M_0 \models o(\kappa) \geq 1$$



By our standard reflection argument, we get that

$$M_0 \models \exists \mu < j(\kappa) \quad o(\mu) \geq 1$$

so $V_\lambda \models \exists \mu < \kappa \quad o(\mu) \geq 1$

So, if $o(\kappa) \geq 2$, then κ is not the least cardinal with $o(\kappa) \geq 1$.

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(3) STRENGTH

Moving away from defining large cardinal notions by ultrafilters and move to talking directly about the embedding:

Let $j : V_\lambda \longrightarrow M$ ← inner model

be any elementary embedding with $\mathrm{crit}(j) = \kappa$.

So, as before $j \restriction V_\kappa = \mathrm{id} \restriction V_\kappa$.

<u>Def.</u> We call j <u>α-strong</u> if

$$V_{\kappa+\alpha} \subseteq M.$$

We have already seen that
if $j_U : V_\lambda \longrightarrow M_U$ is the
ultrapower embedding, then

j_U is 1-strong $\overline{\underline{\text{XII}}}$, p. 8
$V_{\kappa+1} \subseteq M_U$

j_U is not 2-strong $\overline{\underline{\text{XIII}}}$, p. 2
$V_{\kappa+2} \nsubseteq M_U$

<u>Def.</u> A cardinal κ is called
<u>α-strong</u> if there is an α-strong
embedding with critical pt κ.

<u>Observe</u> 1-strong $\Longleftrightarrow$ measurable.

Lecture $\overline{\underline{\text{XV}}}$: 2-strong cardinals.