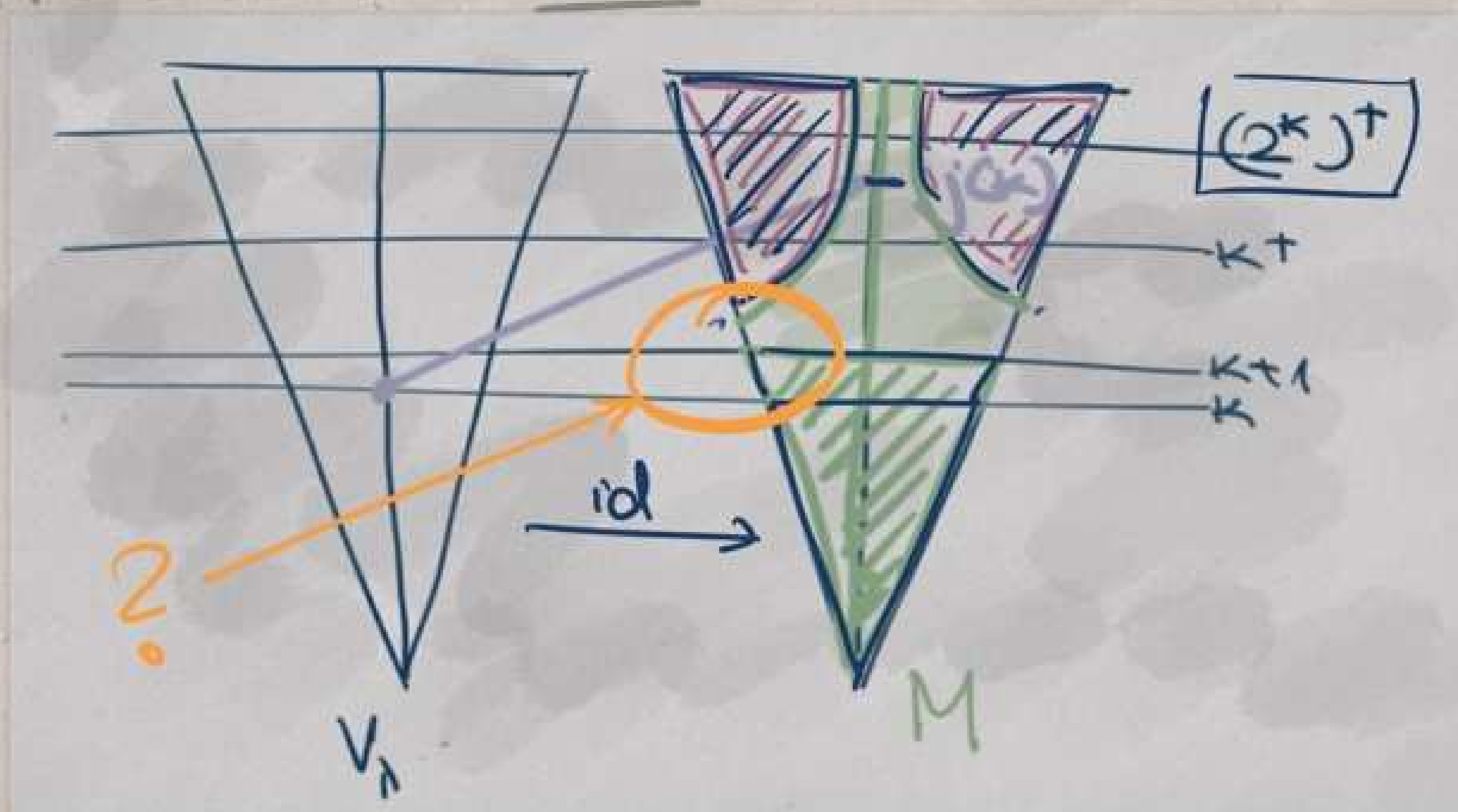
Lecture XIII

Large Cardinals

7 March 2022

FROM LECTURE XII PAGE 11:



First goal: Provide a concrete witness for $V_\lambda \neq M$!

Concretely: is $V_{\kappa+2} \subseteq M$?

Theorem $U \notin M$.

Since $U \in V_{\kappa+2}$, this implies $V_{\kappa+2} \notin M$.

How big is $j(\kappa)$ really?

I claim that $|j(\kappa)| \leq 2^\kappa$.

For this we just count elements of $j(\kappa)$:

if $[f]$ is an ordinal s.t.

$$[f] \in j(\kappa) = [c_\kappa]$$

then w.l.o.g., $f : \kappa \longrightarrow \kappa$.

Thus there are only 2^κ many such functions.

Corollary $j(\kappa)$ is not a strong limit cardinal.

FROM LECTURE $\overline{\text{XII}}$, PAGE 10.

Idea of the proof: use the fact that

$$V_\lambda \models |j(\kappa)| \leq 2^\kappa$$

and show that if U was in M, then the same is true in M. This contradicts $M \models j(\kappa)$ is measurable.

Proof. Suppose towards a contradiction that $U \in M$.

First of all, note that if $f : \kappa \longrightarrow \kappa$, then $f \in V_{\kappa+1}$ and so $f \in M$.

Therefore $\kappa^\kappa \in M$.

Remember that in V_λ,

$$j(\kappa) = \{ (f)_U ; f : \kappa \longrightarrow \kappa \}$$

Thus $f \longmapsto (f)_U$ is a surjection from κ^κ onto $j(\kappa)$. Therefore, it's enough to show that this function exists in M.

Since we assume $U \in M$, the equivalence relation $\sim_U$ on κ^κ defined by

$$f \sim_U g :\Longleftrightarrow \{\alpha ; f(\alpha) = g(\alpha)\} \in U$$

is an element of M by the axiom of separation.

Thus $[f]_U$ is an element of M, again by separation.

But the Mostowski collapse is uniquely defined in models of ZFC, so in M, we can define

[In M define]

$$f \longmapsto (f)_U$$

by $(f)_U$ is the unique image of the Mostowski collapse of $[f]_U$.

Again by seperation

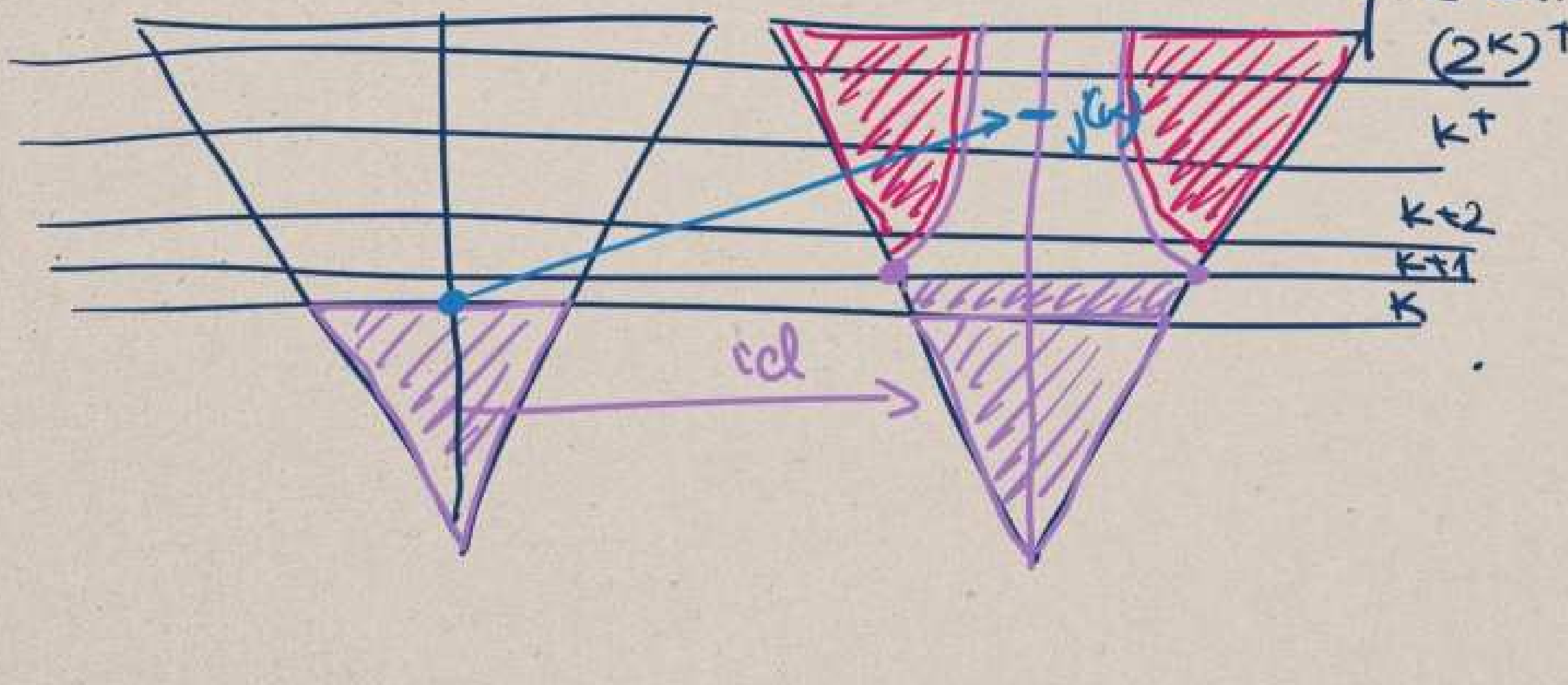
$$f \longmapsto (f)_U$$

is an element of M, so

$$|j(\kappa)| \le 2^\kappa$$

in contradiction to $M \models j(\kappa)$ is measurable.

q.e.d.



<u>Remark</u> This clearly implies that κ cannot be measurable in M via the ultrafilter U.

But it does <u>not</u> exclude the possibility that there is another ultrafilter U' on κ s.t. U' <u>survives</u> in the ultrapower.

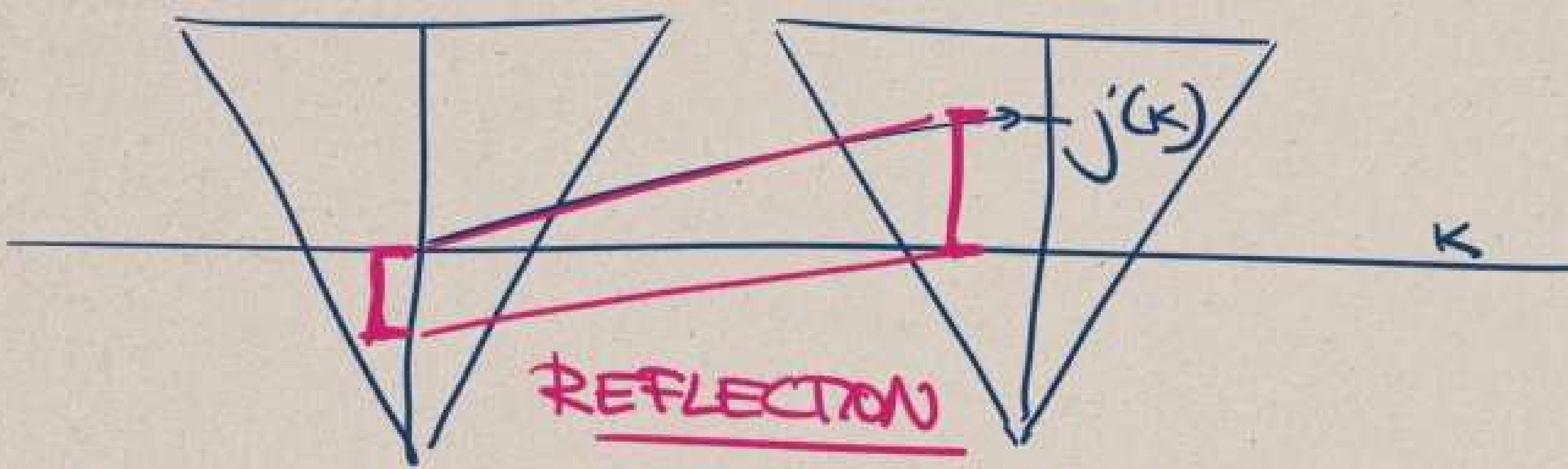
In that case, κ would still be measurable in M.

<u>Def.</u> A cardinal κ is called <u>surviving</u> if it is measurable with ultrapower embedding

$$j : V_\lambda \longrightarrow M$$

and $M \models \kappa$ is measurable.

Compare: surviving vs measurable.



Suppose κ is surviving. Then

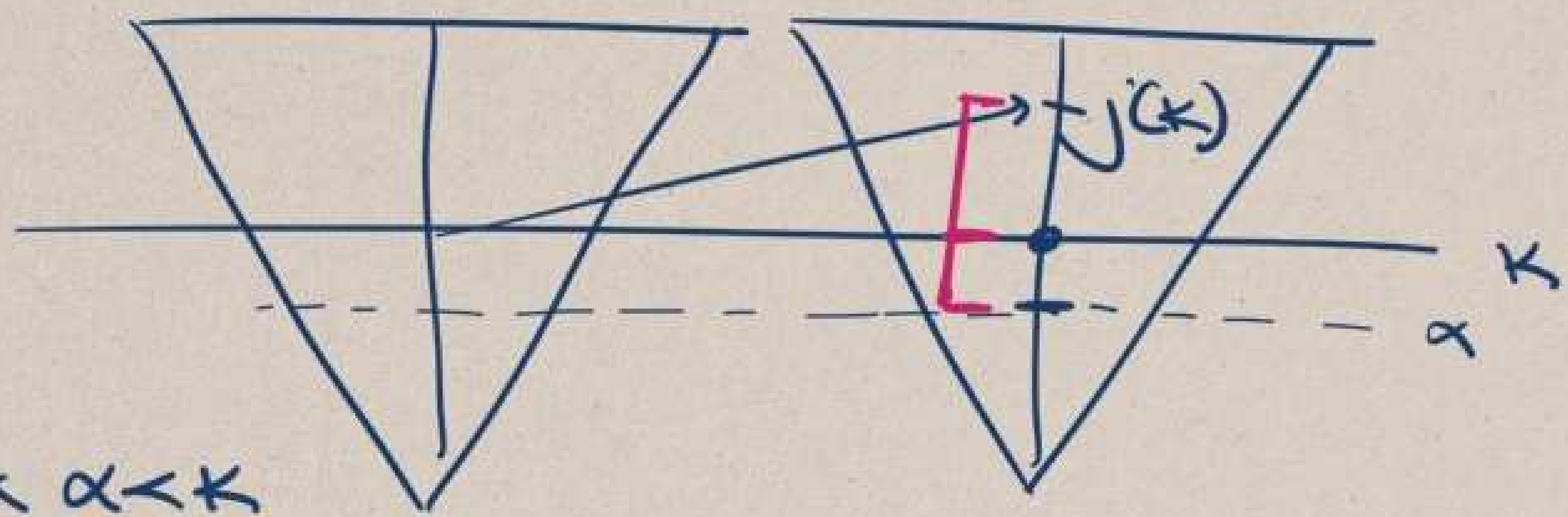
$$M \models \text{there is } \lambda < j(\kappa) \text{ s.t. } \lambda \text{ is measurable}$$

Thus: $$V_\lambda \models \text{there is } \lambda < \kappa \text{ s.t. } \lambda \text{ is measurable.}$$

Corollary If κ is surviving, then κ cannot be the least measurable cardinal.

So, the existence of a surviving cardinal implies the existence of at least two measurables.

IMPROVED RESULT:



Fix $\alpha < \kappa$

$$M \models \exists \beta \, (\underbrace{\alpha}_{= j(\alpha)} < \beta < j(\kappa) \text{ and } \beta \text{ is measurable})$$

$= j(\alpha)$ since $j \restriction V_\kappa = id$.

$$\iff M \models \exists \beta \, (j(\alpha) < \beta < j(\kappa) \text{ and } \beta \text{ is measurable})$$

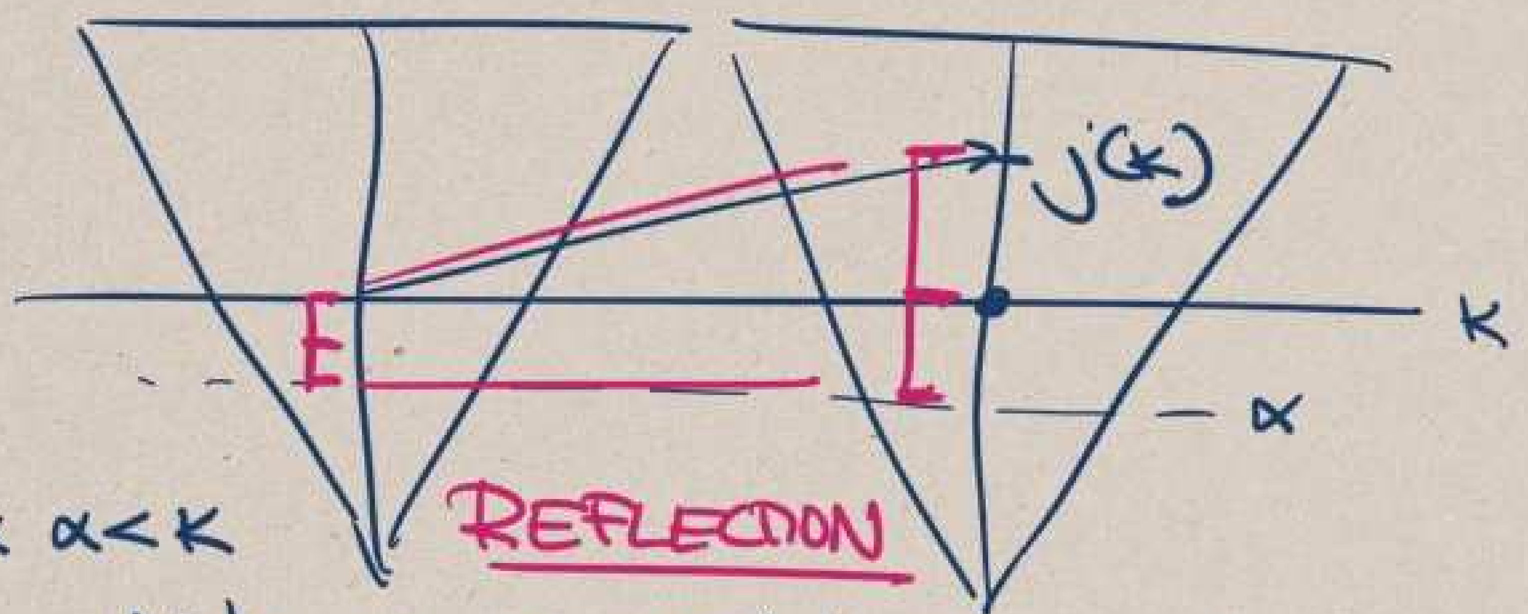
$$\implies V_\lambda \models \exists \beta \, (\alpha < \beta < \kappa \text{ and } \beta \text{ is measurable}).$$

Thus: The set of measurable cardinals below κ is unbounded: a surviving cardinal κ must be the κ-th measurable cardinal.

Let's return to weak compactness.

<u>Q.</u> Is κ weakly compact in M?

If the answer is "yes", we have resolved our question about the relative strength of measurability vs weak compactness:



fix $\alpha<\kappa$

$M \models \kappa$ is weakly compact

$\Longrightarrow M \models \exists \beta \, (j(\alpha) < \beta < j(\kappa) \wedge \beta$ is weakly compact$)$

$\Longrightarrow V_1 \models \exists \beta \, (\alpha < \beta < \kappa \wedge \beta$ is weakly compact$)$

Corollary Assuming $M \models \kappa$ is weakly compact, then κ is the κ-th weakly compact cardinal.

In particular, if α is any ordinal less than the least measurable, then the α-th weakly compact is not measurable.

Theorem $M \models \kappa$ is weakly compact

Proof Let L be an $\mathcal{L}_{\kappa\kappa}$-language with at most κ many non-logical symbols and Φ a set of L-formulas s.t.

$$M \models \Phi \text{ is } \kappa\text{-satisfiable}.$$

So for each $\Phi_0 \subseteq \Phi$ s.t. $|\Phi_0| < \kappa$, there is a model $N_{\Phi_0} \models \Phi_0$ with $N_{\Phi_0} \in M$.

Since $M \subseteq V_\lambda$, $N_{\Phi_0} \in V_\lambda$ and since "$N_{\Phi_0} \models \Phi_0$" is a bounded formula, we get

$V_\lambda \models \Phi$ is κ-satisfiable

Since κ is measurable, κ is weakly compact in V_λ and so

$V_\lambda \models \Phi$ is satisfiable.

i.e., we get some $N \in V_\lambda$ s.t. $N \models \Phi$.

We want that $N \in M$. If so, then again by absoluteness of $N \models \Phi$, we have that $M \models \Phi$ is satisfiable.

In order to prove that $N \in M$, we need to understand better what we can say about N.

In particular, can we put any size bound on N?

[i.e., a LöSko theorem for $\mathcal{L}_{\kappa\kappa}$?]

Let's first explore whether a size bound helps:

Suppose $|N| \leq \kappa$.

This means that we can think of N as a structure on κ:

$$N = (\kappa, \_\_)$$

interpretations of L-symbols

[Relations: subsets of κ^n
Functions: functions from $\kappa^n \to \kappa$
Constants: elements of κ]

This would be something that lives in $V_{\kappa+1}$ and therefore in M.

So, there is an isomorphic copy of N that lives in the part of V_λ that is preserved in M and so M contains that isomorphic copy, and so

$M \models \Phi$ is satisfiable.

Summary

$\mathcal{M} \models \Phi$ is κ-satisfiable

$\Downarrow$

$V_\lambda \models \Phi$ is κ-satisfiable

$\Downarrow$

$V_\lambda \models \Phi$ is satisfiable, say by N

$\Downarrow$ ⟵ SIZE BOUND!

$V_\lambda \models \Phi$ is satisfiable by a model of size $\leq \kappa$

$\Downarrow$

$\mathcal{M} \models \Phi$ is satisfiable.

Remains to show: If Φ is satisfiable, then Φ has a model of size $\leq \kappa$.

Preview: "Strong Lö-Sko for $\mathcal{L}_{\kappa\kappa}$-languages": κ-Skolem hull and Tarski-Vaught-Test.