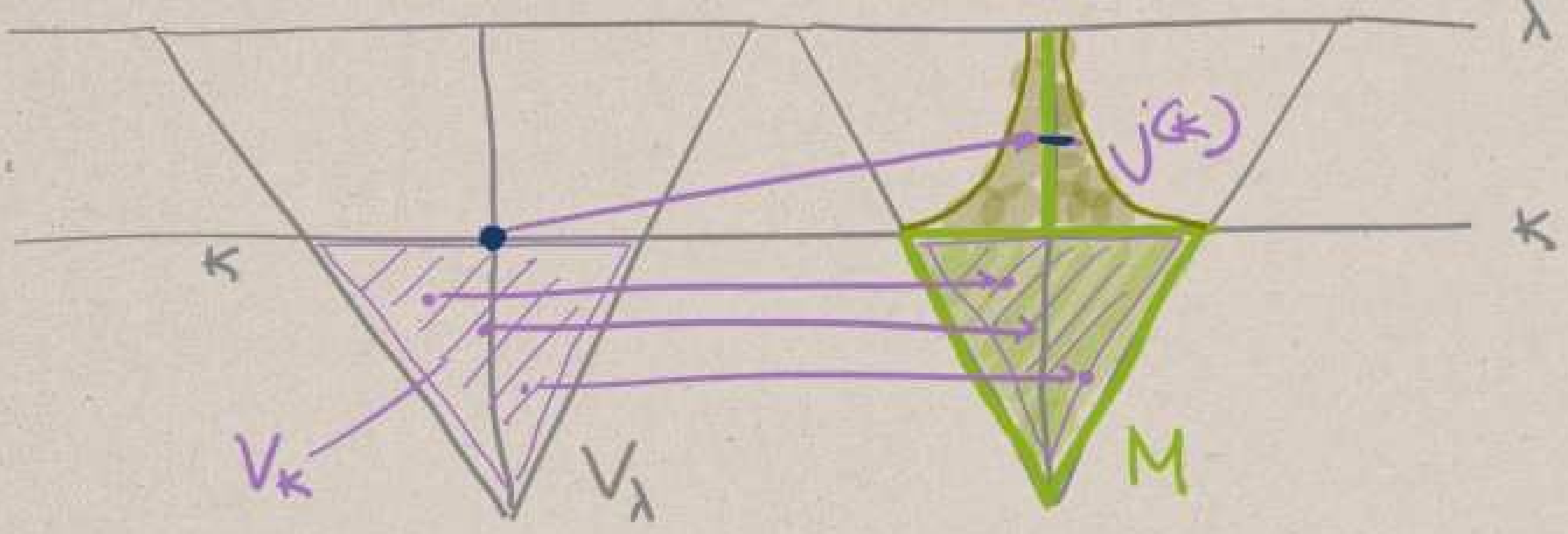
LARGE CARDINALS XII

2 MARCH 2022



$$j_U = j : V_\lambda \longrightarrow M \quad \text{elementary embedding}$$

$$x \longmapsto (c_x)$$

$M \subseteq V_\lambda$ with $\mathrm{Ord} \cap M = \lambda$

$$\boxed{j \upharpoonright V_\kappa = \mathrm{id} \upharpoonright V_\kappa}$$

$$\boxed{\kappa \leq (\mathrm{id}) < j(\kappa)}$$

κ is the critical point of j

FUNDAMENTAL THEOREM ON MEASURABLE CARDINALS

If λ is inaccessible and $\kappa < \lambda$, these TFAE

(i) κ is measurable

(ii) there is a transitive $M \subseteq V_\lambda$ and an elementary embedding $j : V_\lambda \longrightarrow M$ with critical point κ.

<u>Proof.</u> We saw the direction (i) $\Rightarrow$ (ii) in lecture $\overline{\text{XI}}$.

Now (ii) $\Rightarrow$ (i).

First observation: if j is an embedding from V_λ to M with critical point κ

$$[\, \mathrm{crit}(j) = \kappa = \mathrm{c.p.}(j) \,]$$

then $j \restriction V_\kappa = \mathrm{id} \restriction V_\kappa$.

[Induction on rank.]

Define $U \subseteq \mathcal{P}(\kappa)$ as follows:

$$X \in U :\Longleftrightarrow X \subseteq \kappa \text{ and } \kappa \in j(X).$$

$$\boxed{X \in U \iff X \subseteq \kappa \text{ and } \kappa \in j(X)}$$

Claim U is a κ-complete nontrivial uf. on κ.

1. Nontriviality $j(\{\alpha\}) = \{j(\alpha)\} = \{\alpha\}$
by elementarity by $\kappa = \mathrm{crit}(j)$

So for any $\alpha < \kappa$, $\kappa \notin j(\{\alpha\})$.

2. Filter We have, by elementarity,

$$j(X \cap Y) = j(X) \cap j(Y)$$

$$X \subseteq Y \implies j(X) \subseteq j(Y)$$

$$j(\emptyset) = \emptyset$$

So we get directly that

$$X, Y \in U \implies X \cap Y \in U$$

$$X \subseteq Y, X \in U \implies Y \in U$$

$$\emptyset \notin U.$$

Because $\kappa = \mathrm{crit}(j)$, $\kappa < j(\kappa)$, so

$$\kappa \in j(\kappa)$$

$$\implies \kappa \in U.$$

3. <u>Ultra.</u> $j(\kappa \setminus X) = j(\kappa) \setminus j(X)$

by elementarity

So if $\kappa \notin j(X)$, then $\kappa \in j(\kappa \setminus X)$.

4. <u>κ-completeness</u>

Fix some $\boxed{\delta < \kappa}$ and some sequence of sets in U of length δ:

$$A_\alpha \in U \quad \boxed{\alpha < \delta} < \kappa$$

$$[\kappa \in j(A_\alpha)]$$

<u>Want:</u> $\bigcap_{\alpha<\delta} A_\alpha \in U$!

Consider $\mathcal{A} = (A_\alpha ; \alpha < \delta)$.

$V_\lambda \models \mathcal{A}$ is a sequence of subsets of κ of length δ

$\xRightarrow{\text{elem.}}$ $M \models j(\mathcal{A})$ is a seq. of subsets of $j(\kappa)$ of length $j(\delta) = \delta$.

$V_\lambda \models A_\alpha$ is the αth element of $\mathcal{A}$

$\xRightarrow{\text{elem.}}$ $M \models j(A_\alpha)$ is the $j(\alpha)$th element of $j(\mathcal{A})$

$$j(\alpha) = \alpha$$

Summary:

$$j(\vec{A}) = \left(j(A_\alpha);\ \alpha<\delta\right)$$

If $A := \bigcap_{\alpha<\delta} A_\alpha$, then $j(A) = \bigcap_{\alpha<\delta} j(A_\alpha)$

by what we just showed.

Therefore $\kappa \in j(A)$, and so $A \in U$.

q.e.d.

[Remark. There is a remarkable similarity of this proof and the compactness proof of "κ s.c. $\Rightarrow$ κ measurable".]

SOME OBSERVATIONS ABOUT CARDINALS IN M

(1) Since $V_\lambda \models \kappa$ is measurable, we get

$$M \models j(\kappa) \text{ is measurable.}$$

In particular, if $\kappa<\alpha<j(\kappa)$ then there is some μ s.t. $\alpha<\mu<j(\kappa)$ such that

$$M \models \mu \text{ is inaccessible.}$$

These are all between κ and λ.

This is a first hint that M may be not equal to V_λ:
If λ happens to be the least inaccessible above κ, then for every μ s.t.

$$\kappa < \mu < \lambda$$

we have that

$$V_\lambda \models \mu \text{ is not inaccessible.}$$

Since $M \models \mu$ is inaccessible for some of these, we have (in this case) $M \neq V_\lambda$.

(2) Since M is transitive in V_λ, we know that

"cardinal"
"regular"
"inaccessible"

are downwards absolute,

so if $V_\lambda \models \mu$ is a cardinal / regular / inaccessible,

then $M \models \mu$ is a cardinal / regular / inaccessible.

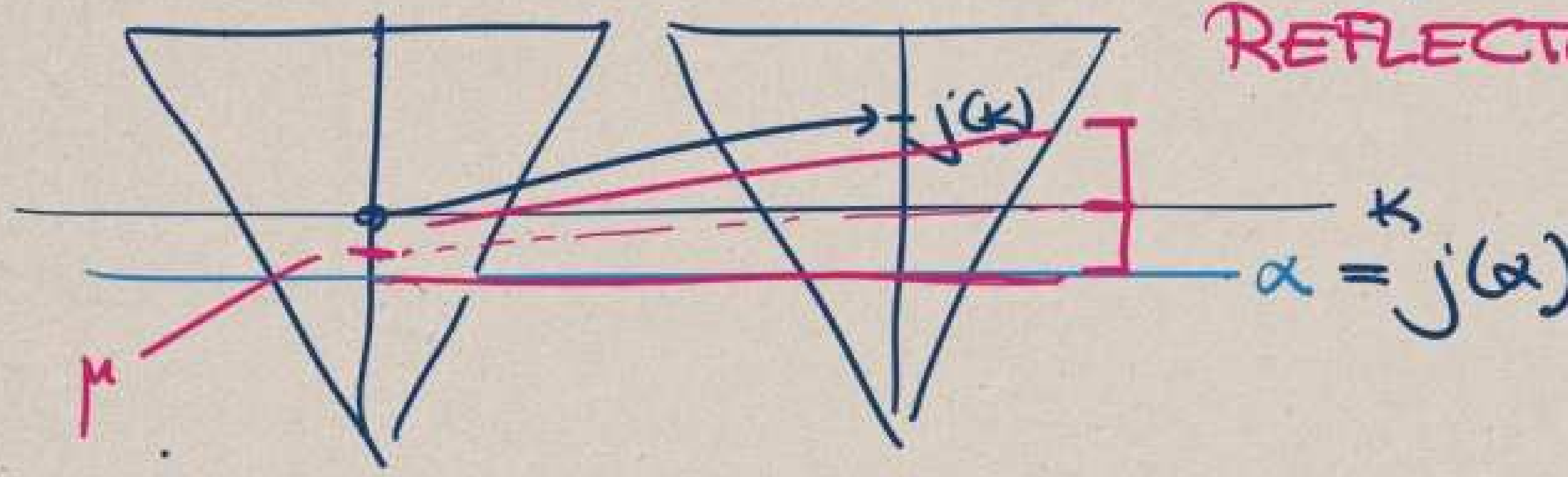
This is certainly true for the inaccessible cardinal κ.

Thus:

$M \models \kappa$ is inaccessible.

This allows us to a reflection argument:

REFLECTION



$M \models$ there is μ s.t.

$j(\alpha) < \mu < j(\kappa)$

μ inaccessible

$\overset{\text{elem.}}{\Longrightarrow}$ $V_\lambda \models$ there is μ s.t.

$\alpha < \mu < \kappa$

μ inaccessible

This yields a new proof of

"If κ is measurable, then there are unboundedly many inaccessibles below it."

Theorem $V_{\kappa+1} \subseteq M$

Proof. Note that we cannot expect

$$j \restriction V_{\kappa+1} = \mathrm{id} \restriction V_{\kappa+1},$$

since $\kappa \in V_{\kappa+1}$, but $j(\kappa) \neq \kappa$.

If $X \in V_{\kappa+1}$, then $\boxed{X \subseteq V_\kappa}$.

Claim $X = j(X) \cap V_\kappa$.

[This claim implies that $X \in M$ and finishes the proof of the thm.]

"$\subseteq$": If $x \in X$, then $x \in V_\kappa$. Thus $j(x) = x$.

$x \in X \xrightarrow{\text{elem.}} j(x) \in j(X)$

$$\left.\begin{array}{l} x \in V_\kappa,\ j(x)=x \\ j(x) \in j(X) \end{array}\right\} x \in j(X) \cap V_\kappa.$$

"$\supseteq$": Let $x \in j(X) \cap V_\kappa$, so

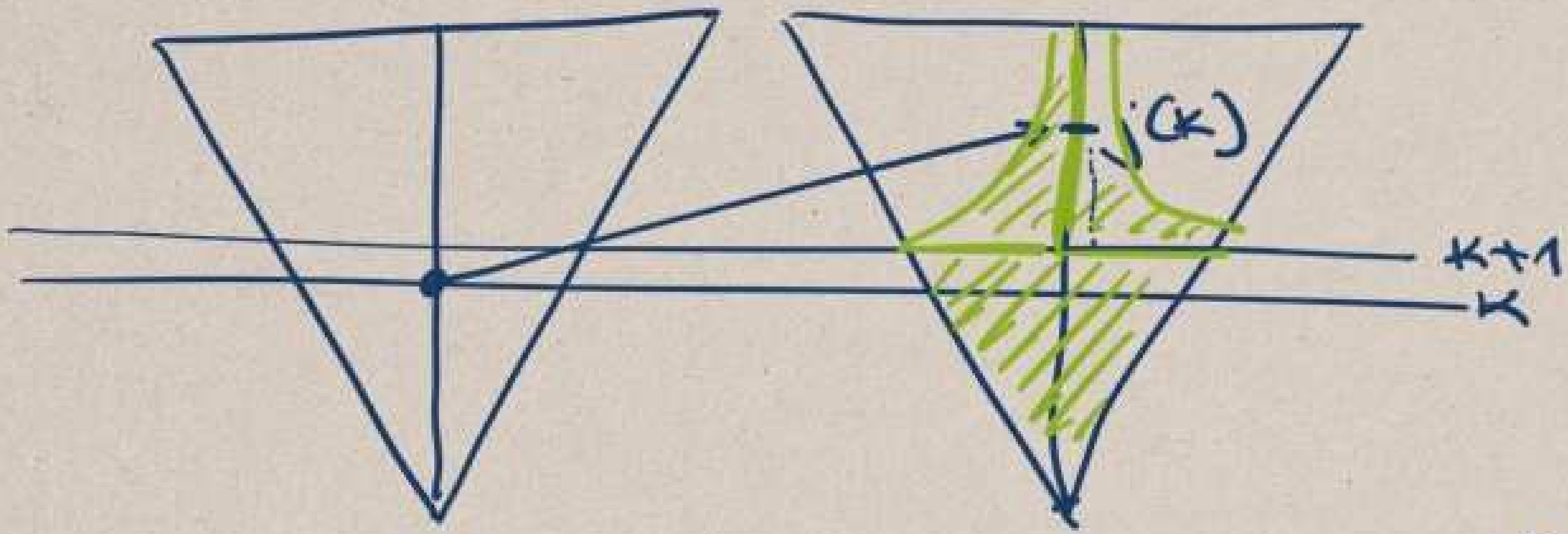
$$\left.\begin{array}{l} x \in j(X) \\ \& \\ x \in V_\kappa \longrightarrow x = j(x) \end{array}\right\} j(x) \in j(X)$$

$$\Downarrow \text{elem.}$$

$$x \in X.$$

This proves the claim.

q.e.d.



Corollary. If κ^+ is the successor of κ in V_λ (in reality), then

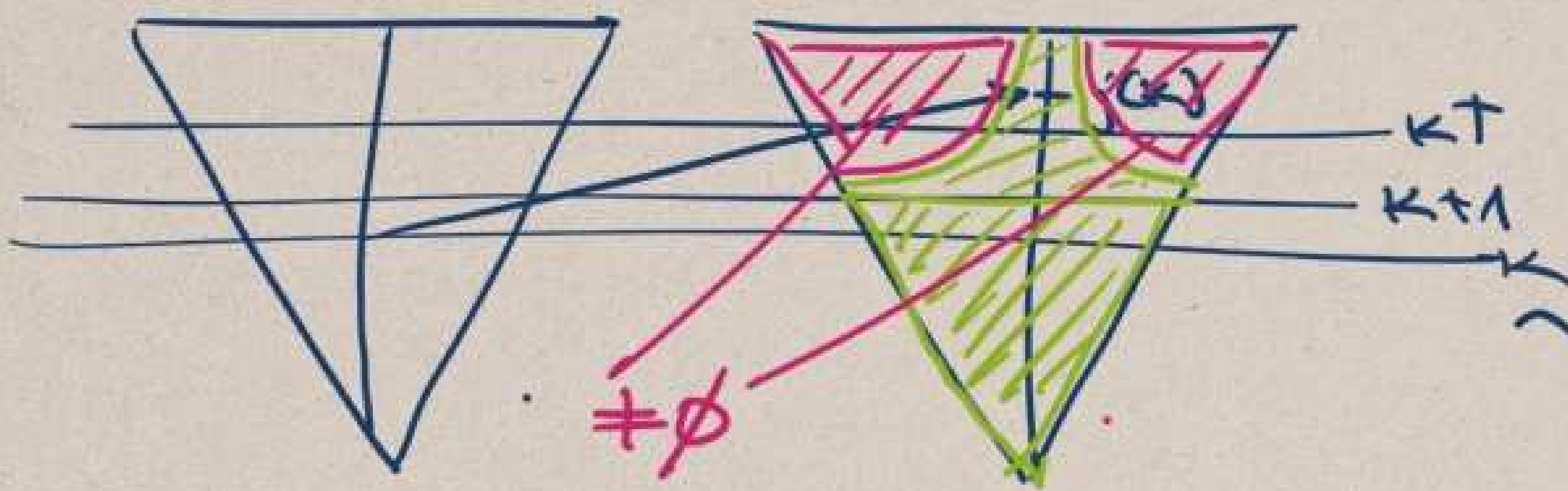
$M \models \kappa^+$ is the successor of κ

[i.e., if $\kappa < \alpha < \kappa^+$, then α is not a cardinal in M.]

Proof. Suppose $\kappa < \alpha < \kappa^+$. Then there is a wellorder of order type α on κ. This wellorder is a subset of V_κ, so an element of $V_{\kappa+1}$, so (by Thm) an element of M.

Thus $M \models \alpha$ is not a cardinal.

q.e.d.



How big is $j(\kappa)$ really?

I claim that $|j(\kappa)| \leq 2^{\kappa}$.

For this we just count elements of $j(\kappa)$:

If (f) is an ordinal s.t.

$$(f) \in j(\kappa) = (c_\kappa)$$

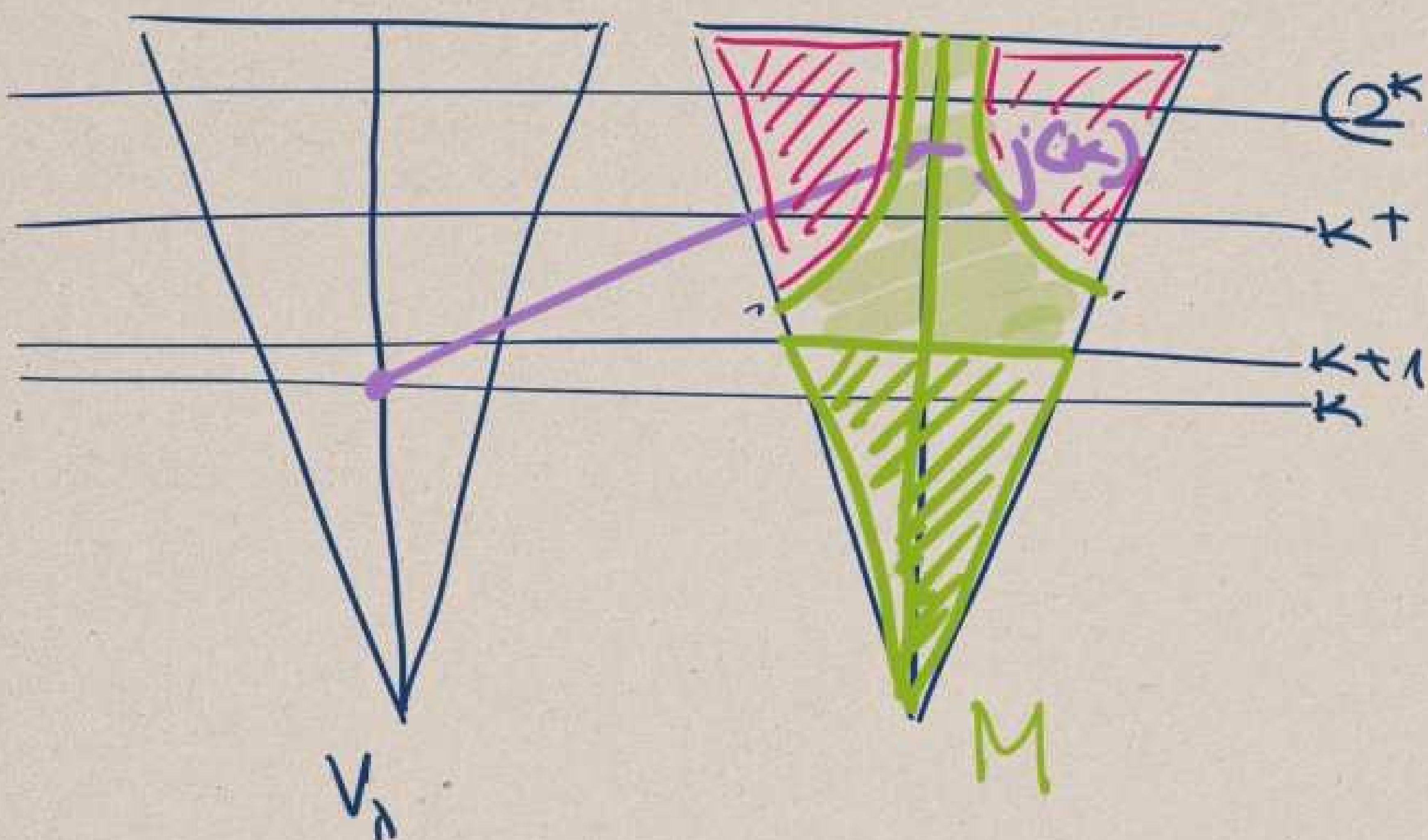
then w.l.o.g., $f : \kappa \longrightarrow \kappa$.

Thus there are only 2^{κ} many such functions.

Corollary $j(\kappa)$ is not a strong limit ordinal.

Corollary $j(\kappa)$ is not inaccessible

Corollary $V_\lambda \neq M$.



Next goal: Produce a concrete example of a set in $V_\lambda \setminus M$.

· This will be the set U.