Eleventh Lecture: LARGE CARDINALS

28 February 2022

ASIDE: Triviality of filters

- A filter F is called free if $\bigcap F = \emptyset$ and fixed if it is not free.
- We said that a filter is nontrivial (Lecture VI, page 5) if it does not contain singletons.
- If U is an ultrafilter, then U is free iff U is nontrivial.
- This is not in general true for filters. [⟶ Example Sheet #3]

REMINDER $\kappa < \lambda$, κ measurable, λ inaccessible, U κ-complete nontrivial uf. on κ

$f : \kappa \longrightarrow V_\lambda$

$X := \mathrm{Ult}(V_\lambda, U)$ can be considered as a subset of V_λ.

$$V_\lambda \longrightarrow X \qquad \text{elementary embedding}$$
$$x \longmapsto [c_x]$$

$(X, E) \models$ Extensionality

The analogue of 'subset collapse' is:

LEADER, Logic & Set Theory NOTES, §5

Theorem 4 (Mostowski's Collapsing Theorem). Let r be a relation on a set a that is well-founded and extensional. Then there exists a transitive set b, and a bijection $f : a \to b$ such that $(\forall x, y \in a)(x \; r \; y \Leftrightarrow f(x) \in f(y))$. Moreover, b and f are unique.

Remark. 'Well-founded' and 'extensional' are trivially necessary.

Want: X is extensional and wellfounded

Theorem (X, E) is wellfounded.

Proof. Suppose not, then there is a sequence f_n of functions decreasing in E.

$$[f_{n+1}] \, E \, [f_n]$$

$$\Longleftrightarrow \{\alpha ; f_{n+1}(\alpha) \in f_n(\alpha)\} \in U.$$

$A_n :=$

So, by κ-completeness, we have

$$A := \bigcap_{n \in \mathbb{N}} A_n \in U$$

$$= \{\alpha ; \forall n \; f_{n+1}(\alpha) \in f_n(\alpha)\}$$

Since $A \in U$, $A \neq \emptyset$, so pick $\alpha \in A$ and get $f_{n+1}(\alpha) \in f_n(\alpha)$ for all n.

That is in contradiction to the wellfoundedness of V_λ. q.e.d.

Remarks

1. Only needed $\aleph_1$-completeness.
2. Proved "nonexistence of decreasing sequences" which is only eq. to wellfoundness assuming AC.
3. $\aleph_1$-completeness is equivalent to nonex. of decr. seq. (Example Sheet #3).

Using wellfoundness + Mostowski, we can collapse X to a transitive set M s.t.

$$\pi : (X, E) \cong (M, \in)$$

Notation: $f : \kappa \longrightarrow V_\lambda$

$$(f)_U := \pi([f]_U)$$

[omitting U if clear].

Concatenating the two embeddings, we get

$$j := j_U : V_\lambda \longrightarrow M$$

$$x \longmapsto (c_x)_U .$$

<u>Claim 1</u> $|M| = \lambda$.

[$\{[c_\alpha]; \alpha < \lambda\} \subseteq X \subseteq V_\lambda$, thus

$\lambda = |\{[c_\alpha]; \alpha < \lambda\}| \leq |X| \leq |V_\lambda| = \lambda$

(because λ is inaccessible)

So $|X| = \lambda$, so $|M| = \lambda$, since $(M, \in) \cong (X, E)$.]

<u>Claim 2</u> If $x \in M$, then $|x| < \lambda$.

[Suppose $x \in M$, so $f : \kappa \longrightarrow V_\lambda$ s.t. $x = \pi(f)$. By regularity of λ, we find α s.t. $f : \kappa \longrightarrow V_\alpha$.

If $y \in x$, then by transitivity of M, $y \in M$, so there is $g : \kappa \longrightarrow V_\lambda$ s.t. $\pi(g) = y$.

Since $y \in x$, we have $\pi(g) \in \pi(f)$, so $[g] E [f] \iff \{\alpha ; g(\alpha) \in f(\alpha)\} \in U$.

So, w.l.o.g. $g : \kappa \longrightarrow V_\alpha$

Therefore

$$|x| \leq |V_\alpha|^\kappa.$$

Since λ is inaccessible, $|V_\alpha| < \lambda$ [Hausdorff's Theorem / Proof], so

$$|x| < \lambda. \;]$$

<u>Claim 3</u> $M \subseteq V_\lambda$.

[By Claim 2., all elements of M are small; since M is transitive, this means $|\mathrm{tcl}(x)| < \lambda$ if $x \in M$, so $M \subseteq H_\lambda = V_\lambda$, by ES#1 (7).]

<u>Claim 4</u> $\mathrm{Ord} \cap M = \lambda$

[Remember $\{[c_\alpha] ; \alpha < \lambda\} \subseteq X$ and $\{(c_\alpha) ; \alpha < \lambda\} \subseteq M$ which is a set of ordinals of o.t. λ. So

$$\lambda \leq \mathrm{Ord} \cap M \underset{\text{Claim 3.}}{\leq} \lambda \;]$$

Remark If $f : \kappa \longrightarrow \lambda$, then (f) is an ordinal.

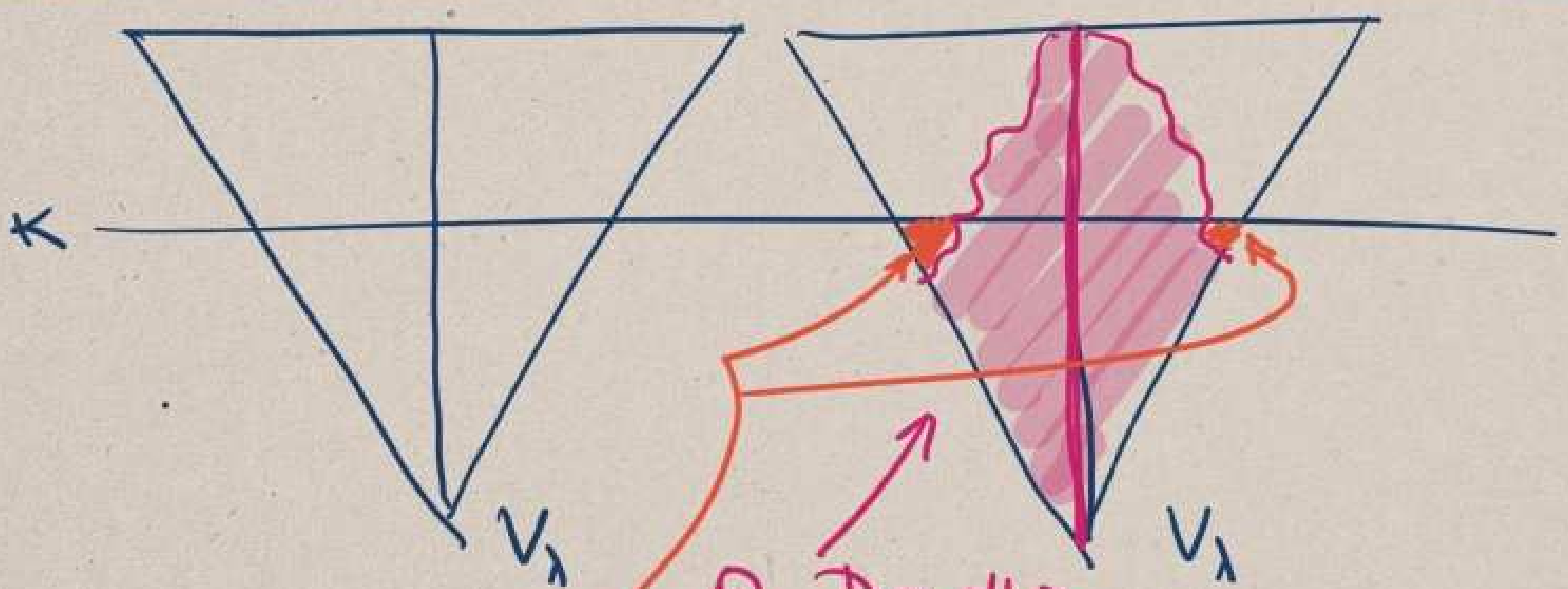
$\kappa = \{\alpha ; f(\alpha) \text{ is an ordinal}\} \in U$

$(X, E) \models [f]$ is an ordinal

$\Downarrow$

$(M, \in) \models (f)$ is an ordinal

Being an ordinal is absolute for transitive models of ZFC.



Q. Does this picture really look like this?

[NO!]

We'll show that these corners can't exist.

Claim 5 $\quad j \restriction V_\kappa = \mathrm{id}$

[Proof by $\in$-induction on V_κ.
Suppose $x \in V_\kappa$ arbitrary s.t. the claim is true for all $y \in x$.

$$y \in x \iff (M,\in) \models j(y) \in j(x)$$
$$\overset{IH}{\iff} (M,\in) \models y \in j(x)$$
$$\iff y \in j(x)$$

If $y \in x$, then $y \in j(x)$.
Thus $x \subseteq j(x)$. $\quad (c_x) = j(x)$

If now $f : \kappa \longrightarrow V_\lambda$ s.t. $(f) \in j(x)$,
then $\{\alpha ; f(\alpha) \in c_x(\alpha)\} \in U$

$$\{\alpha ; f(\alpha) \in x\} = \bigcup_{y \in x} \{\alpha ; f(\alpha) = y\}$$

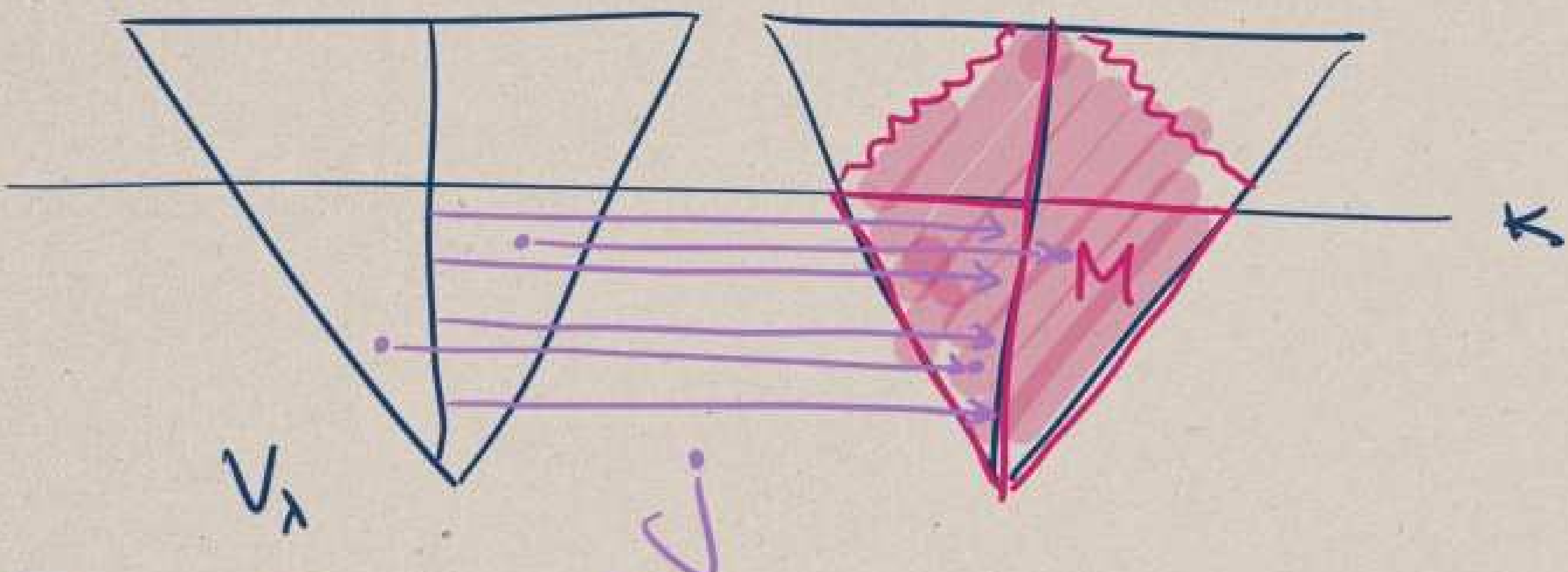
$\Longrightarrow$ there is $y \in x$ s.t. $\{\alpha ; f(\alpha) = y\} \in U$.
$\Longrightarrow (f) = (c_y) = j(y) \overset{IH}{=} y$.

This is a union of size $|x| < \kappa$, since κ is measurable, hence inaccessible.

$\Longrightarrow j(x) \subseteq x.$

Together: $j(x) = x.$]

Improve picture



<u>Claim 6</u> $j(\kappa) > \kappa.$

[As seen before, $j(\kappa)$ is an ordinal.
Also, clearly $\kappa \leq j(\kappa)$.
We prove the claim by providing an ordinal that sits between all $\alpha < \kappa$ and $j(\kappa)$.

Consider $id : \kappa \longrightarrow V_\lambda$.
By previous remark (id) is an ordinal.

Fix $\gamma < \kappa$ and compare (id) with γ:

$$A_\gamma := \{\alpha ; id(\alpha) > c_\gamma(\alpha)\}$$
$$= \{\alpha ; \alpha > \gamma\} = \kappa \setminus \gamma$$
$$\in U$$

Therefore $(id) > (c_\gamma) = j(\gamma)$
$$\underset{\text{Claim 5}}{=} \gamma.$$

This shows $\boxed{(id) \geq \kappa}$ (*)

[Since $\gamma < \kappa$ was arbitrary.]

Also consider

$$\{\alpha ; id(\alpha) < c_\kappa(\alpha)\}$$
$$= \{\alpha \in \kappa ; \alpha < \kappa\} = \kappa \in U$$

So $\boxed{(id) < (c_\kappa) = j(\kappa).}$ (**)

By (*) and (**), we get

$$\kappa \leq (id) < j(\kappa). \quad]$$

Remark An embedding $j: V_\lambda \longrightarrow M$ is called nontrivial if $j \neq \mathrm{id}$.

We can show (ES #3) that for every nontrivial embedding there is an ordinal γ s.t. $j(\gamma) > \gamma$.

The least such ordinal is called the critical point of j.

Theorem (Fundamental Theorem on Measurable Cardinals)

Suppose λ is inaccessible and $\kappa < \lambda$. Then TFAE:

(i) κ is measurable

(ii) There is a transitive $M \subseteq V_\lambda$ and an nontrivial elem. embedding

$$j: V_\lambda \longrightarrow M$$

with critical point κ.

We just proved (i) $\Longrightarrow$ (ii).

In Lecture XII, (ii) $\Longrightarrow$ (i).