Large Cardinals X

Tenth Lecture

23 February 2022

Logical Hierarchies

Consequence hierarchy.

First attempt:

$\Phi \leq_0 \Psi$ iff $C_\Phi \subseteq C_\Psi$

where $C_\Phi := \{\varphi \,;\, \Phi \vdash \varphi\}$

$\Phi \equiv_0 \Psi$ iff $\Phi \leq_0 \Psi \leq_0 \Phi$

$\Phi <_0 \Psi$ iff $\Phi \leq_0 \Psi$ and $\Phi \not\equiv_0 \Psi$

Problem: $\mathrm{ZFC} + \mathrm{CH} \not\leq_0 \mathrm{ZFC} + \neg\mathrm{CH}$ and $\mathrm{ZFC} + \neg\mathrm{CH} \not\leq_0 \mathrm{ZFC} + \mathrm{CH}$

Second attempt: If Φ is a cardinal property we write ΦC for $\exists \kappa\, \Phi(\kappa)$.

If ΦC, then let ι_Φ be the least such cardinal.

So, $\left.\begin{matrix}\iota_I \\ \iota_W \\ \iota_M\end{matrix}\right\}$ is the least $\left\{\begin{matrix}\text{inaccessible} \\ \text{weakly compact} \\ \text{measurable}\end{matrix}\right.$

$\mathrm{ZFC} + \Phi C \leq_1 \mathrm{ZFC} + \Psi C$ iff $\boxed{\Phi C + \Psi C \Longrightarrow \iota_\Phi \leq \iota_\Psi.}$

Make ZFC into a "large cardinal axiom" by ES#1 (3):

$\mathrm{Inf}(\kappa) :\Longleftrightarrow \kappa$ is infinite

$$\mathrm{ZFC} \Longleftrightarrow \mathrm{ZFC} + \mathrm{InfC}$$

$$\mathrm{ZFC}+\mathrm{InfC} <_1 \mathrm{ZFC}+\mathrm{IC} <_1 \mathrm{ZFC}+\mathrm{WC}$$

$$\leq_1 \mathrm{ZFC}+\mathrm{MC}$$

$$\leq_1 \mathrm{ZFC}+\mathrm{SC}$$

We'll improve this to $<_1$ in the next lectures.

Silly examples

(1) $\Sigma(\kappa) :\Longleftrightarrow \exists\lambda\ \lambda$ is weakly compact and κ is inaccessible

$$\mathrm{ZFC}+\Sigma\mathrm{C} \Longleftrightarrow \mathrm{ZFC}+\mathrm{WC}$$

in particular $C_{\mathrm{ZFC}+\Sigma\mathrm{C}} = C_{\mathrm{ZFC}+\mathrm{WC}}$

$$\mathrm{ZFC}+\Sigma\mathrm{C} \equiv_0 \mathrm{ZFC}+\mathrm{WC}.$$

However, if $\Sigma\mathrm{C}+\mathrm{WC}$ hold, then $\iota_\Sigma = \iota_I < \iota_W$, thus $\mathrm{ZFC}+\Sigma\mathrm{C} <_1 \mathrm{ZFC}+\mathrm{WC}$.

② $\Sigma'(\kappa) :\Longleftrightarrow$
$$\Big[\neg WC \Longrightarrow \exists\alpha\,(\aleph_\alpha \text{ is inaccessible } \& \ \kappa = \aleph_{\alpha+\omega})$$

and

$$WC \Longrightarrow \exists\alpha\,(\aleph_\alpha \text{ is weakly compact } \& \ \kappa = \aleph_{\alpha+\omega})\Big]$$

$$ZFC + \Sigma'C \Longleftrightarrow ZFC + IC$$

$$\Longrightarrow \quad C_{ZFC+\Sigma'C} = C_{ZFC+IC}$$

$$\Longrightarrow \quad ZFC + \Sigma'C \equiv_0 ZFC + IC.$$

$$\Sigma'C + WC \Longrightarrow \iota_W < \iota_{\Sigma'}.$$

So $\quad ZFC + WC <_1 ZFC + \Sigma'C.$

<u>Conclusion</u> Neither $\leq_0$ nor $\leq_1$ work nicely: need something that behaves like $\leq_0$ without too much incomparability.

IDENTITY CRISES

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HOW LARGE IS THE FIRST STRONGLY COMPACT CARDINAL?
or
A STUDY ON IDENTITY CRISES

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It is proved that if strongly compact cardinals are consistent, then it is consistent that the first such cardinal is the first measurable. On the other hand, if it is consistent to assume the existence of supercompact cardinal, then it is consistent to assume that it is the first strongly compact cardinal.

An IDENTITY CRISIS is a situation where we have two large cardinal axioms ΦC and ΨC where ΦC is clearly stronger ΨC, but

$$\Phi C + \Psi C \wedge L_{\Phi} = L_{\Psi}$$

is consistent.

The consistency strength hierarchy.

For the sake of simplicity, we focus on theories of the type

$$\mathrm{ZFC} + A$$

where A is a single axiom.

Consider the set of consistency statements

$$\mathrm{Cons} := \{ \mathrm{Cons}(\mathrm{ZFC}+A) ;\ A \text{ is a formula} \}$$

These formulas are all (modulo coding) arithmetical formulas, i.e., all quantifiers are bounded by $\mathbb{N}$.

$\Longrightarrow \Delta_0$

Thus, all consistency statements are absolute between transitive models of set theory.

$$A \leq_{\mathrm{Cons}} B \;:\Longleftrightarrow\; \mathrm{Cons} \cap C_{\mathrm{ZFC}+A} \subseteq \mathrm{Cons} \cap C_{\mathrm{ZFC}+B}$$

Does this remove the problems?

① Σ, Σ' problems go away, since

$$C_{ZFC+\Sigma C} = C_{ZFC+WC}$$

$$\Longrightarrow \mathrm{Cons} \cap C_{ZFC+\Sigma C} = \mathrm{Cons} \cap C_{ZFC+WC}$$

$$\Longrightarrow \quad WC \equiv_{\mathrm{Cons}} \Sigma C$$

Similarly, $IC \equiv_{\mathrm{Cons}} \Sigma' C$.

② Note that our proof of, say,

$$IC <_0 WC$$

proved $ZFC + WC \vdash \mathrm{Cons}(ZFC + IC)$,

and so we still retain

$$ZFC <_{\mathrm{Cons}} ZFC + IC <_{\mathrm{Cons}} ZFC + WC.$$

③ Non-linearity

Both Gödel 1938 [proof of consistency of CH] and Cohen's method of forcing [proof of consistency of ¬CH] produce TRANSITIVE models of CH | ¬CH, respectively.

Therefore, any consistency statement true before application of the method will be true after the application of the method.

This implies that

$$\mathrm{Cons} \cap C_{ZFC} = \mathrm{Cons} \cap C_{ZFC+CH}$$

$$= \mathrm{Cons} \cap C_{ZFC+\neg CH},$$

so

$$ZFC \equiv_{\mathrm{Cons}} ZFC+CH \equiv_{\mathrm{Cons}} ZFC+\neg CH.$$

Non-linearity of the consistency strength hierarchy

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NONLINEARITY IN THE HIERARCHY OF LARGE CARDINAL CONSISTENCY STRENGTH

JOEL DAVID HAMKINS

ABSTRACT. I present instances of nonlinearity and illfoundedness in the hierarchy of large cardinal consistency strength—as natural or as nearly natural as I can make them—and consider philosophical aspects of the question of naturality with regard to this phenomenon. I provide various cautious enumerations of the ZFC axioms which succeed in enumerating all the ZFC axioms, but with a strictly weaker consistency strength than the usual (incautious) enumeration. And similarly there are various cautious versions of many large cardinal hypothesis, which are natural but also incomparable in consistency strength.

It is a mystery often mentioned in the foundations of mathematics, a fundamental phenomenon to be explained, that our best and strongest mathematical theories seem to be linearly ordered and indeed well-ordered by consistency strength. Given any two of the familiar large cardinal hypotheses, for example, generally one of them will prove the consistency of the other.

Why should it be linear? Why should the large cardinal notions line up like this, when they often arise from completely different mathematical matters? Measurable cardinals arise from set-theoretic issues in measure theory; Ramsey cardinals generalize ideas in graph coloring combinatorics; compact cardinals arise with compactness properties of infinitary logic. Why should these disparate considerations lead to principles that are linearly related by direct implication and consistency strength?

Rosser sentences

Theorem 2. *There are statements σ and τ in the language of arithmetic with incomparable consistency strengths over* PA. *That is, neither* $\mathrm{Con}(\mathrm{PA} + \sigma)$ *nor* $\mathrm{Con}(\mathrm{PA} + \tau)$ *provably implies the other over* PA.

Proof. Using the double version of the fixed-point lemma, we can find distinct sentences σ and τ, each asserting that for any refutation of the other sentence in the theory $\mathrm{PA} + \mathrm{Con}(\mathrm{PA})$, there is a smaller refutation of itself, one with a smaller Gödel code. That is, σ asserts that for any proof of $\neg\tau$ in $\mathrm{PA} + \mathrm{Con}(\mathrm{PA})$, there is a smaller proof of $\neg\sigma$; and similarly vice versa with τ.

Neither of these sentences, I claim, is actually refutable in $\mathrm{PA}+\mathrm{Con}(\mathrm{PA})$, since if one of them were refutable, then one of them would have the smallest refutation, and this would make that sentence also provably true in PA, which would contradict the consistency of the theory $\mathrm{PA} + \mathrm{Con}(\mathrm{PA})$. So neither sentence is actually refutable and hence both are (vacuously) true.

Since σ is not refutable, it follows that $\mathrm{PA}+\mathrm{Con}(\mathrm{PA})+\sigma$ is consistent, and so it is also consistent with the assertion of its own inconsistency $\neg\,\mathrm{Con}(\mathrm{PA}+\mathrm{Con}(\mathrm{PA})+\sigma)$. In any model of this combined theory, σ is refutable in $\mathrm{PA}+\mathrm{Con}(\mathrm{PA})$, but since also σ is true there, there must not be any smaller refutation of τ. Since this syntactic situation will be provable in PA, it follows in light of what the sentences assert that the model thinks that PA proves that σ is true and τ is false. So from $\mathrm{Con}(\mathrm{PA})$ it follows both that $\mathrm{Con}(\mathrm{PA} + \sigma)$ and $\neg\,\mathrm{Con}(\mathrm{PA} + \tau)$ in this model.

Similarly, since τ is not refutable, we may consider the theory $\mathrm{PA} + \mathrm{Con}(\mathrm{PA}) + \tau$ analogously, and thereby find a model in which $\mathrm{Con}(\mathrm{PA}+\tau)$ but $\neg\,\mathrm{Con}(\mathrm{PA}+\sigma)$. So the two sentences have incomparable consistency strength over PA, as claimed. $\square$

MEASURABLE CARDINALS, ELEMENTARY EMBEDDINGS & REFLECTION

For the sake of simplicity, we first work under the assumption that there are

$$\kappa < \lambda$$

s.t. κ is measurable and λ is inaccessible.

In particular, $\boxed{V_\lambda \models ZFC + MC}$,

so our assumption are strictly stronger than ZFC+MC (in the sense of $\leq_{\text{Cons}}$).

[Later, we remove the additional inaccessible.]

Fix U a κ-complete nontrivial ultrafilter on κ.

Using our ultrapowers from Lecture $\overline{\text{VIII}}$, we can form

$$\text{Ult}(V_\lambda, U)$$

Reminder

$$\mathrm{Ult}(V_\lambda, U)$$

consists of functions $f : \kappa \longrightarrow V_\lambda$

where $f \sim_U g :\Longleftrightarrow$

$$\{\alpha \in \kappa ; f(\alpha) = g(\alpha)\} \in U.$$

If f is such a function, then $\mathrm{ran}(f)$ is bounded in V_λ (by regularity of λ) and therefore $f \in V_\lambda$.

Thus if we pick for each $\sim_U$-equivalence class a representative and call this set X, then $X \subseteq V_\lambda$.

However, it comes with the following $\in$ relation:

$$f E g :\Longleftrightarrow \{\alpha \in \kappa ; f(\alpha) \in g(\alpha)\} \in U.$$

Clearly, $E \neq \in$.

Summary:

Find $X \subseteq V_\lambda$ and $E \subseteq X \times X$, such that the map

$x \longmapsto c_x$ where c_x is the constant function with value x

is an elementary embedding from $(V_\lambda, \in)$ into (X, E).

$\models ZFC$ $\rightsquigarrow$ $(X, E) \models$ Extensionality

Would like to have some transitive set $M \subseteq V_\lambda$ s.t.

$$(M, \in) \cong (X, E)$$

Call the iso π.

and then

$$\begin{array}{ccccc} V_\lambda & \longrightarrow & X & \longrightarrow & M \\ x & \longmapsto & c_x & \longmapsto & \pi^{-1}(c_x) \end{array}$$

Use MOSTOWSKI:

The analogue of 'subset collapse' is:

LEADER, Logic & Set Theory
NOTES, §5

Theorem 4 (Mostowski's Collapsing Theorem). Let r be a relation on a set a that is well-founded and extensional. Then there exists a transitive set b, and a bijection $f : a \to b$ such that $(\forall x, y \in a)(x \; r \; y \Leftrightarrow f(x) \in f(y))$. Moreover, b and f are unique.

Remark. 'Well-founded' and 'extensional' are trivially necessary.

1. Extensionality comes for free since
$V_\lambda \models$ Extensionality,
so
$(X, E) \models$ Extensionality.
2. Wellfoundedness requires a proof.
[This will use the measurability of κ.]