Large Cardinals

Eighth Lecture

16 February 2022

STRONGLY COMPACT

$\Downarrow$ $\Downarrow$ Keisler-Tarski

MEASURABLE

$\Downarrow$ $\Downarrow$ B

WEAKLY COMPACT

$\Downarrow$ $\Downarrow$ A $\quad \not\Uparrow \leftarrow$ Next Monday IX

INACCESSIBLE

A

<u>Theorem</u> weakly compact $\Longrightarrow$ inaccessible

<u>Proof.</u> We already know "regular".

So need to show strong limit.

Towards a contradiction, assume $\boxed{2^\lambda \geq \kappa}$ for some $\lambda < \kappa$.

$\mathcal{L}_{\kappa\kappa}$ language with constant symbols

$$c_\alpha, \; c_\alpha^0, \; c_\alpha^1 \quad \text{for } \alpha < \lambda.$$

[This is OK for weak compactness.]

$$\Phi^* \left[\begin{array}{l} \Phi \left[\begin{array}{l} \bigwedge_{\alpha<\lambda} c_\alpha^0 \neq c_\alpha^1 \\ \bigwedge_{\alpha<\lambda} c_\alpha = c_\alpha^0 \vee c_\alpha = c_\alpha^1 \end{array} \right. \\ \varphi_f := \bigvee_{\alpha<\lambda} c_\alpha \neq c_\alpha^{f(\alpha)} \quad \text{for } f : \lambda \to 2 \end{array} \right.$$

If $\mathfrak{M} \models \Phi$, we can define a function

$$g : \lambda \longrightarrow 2$$

by

$$g(\alpha) := \begin{cases} 0 & \text{if } \mathfrak{M} \models c_\alpha = c_\alpha^0 \\ 1 & \text{if } \mathfrak{M} \models c_\alpha = c_\alpha^1 . \end{cases}$$

Φ^* is inconsistent since the additional formula says that no such function can exist.

To conclude our contradiction, we only need to show that Φ^* is κ-satisfiable.

$\Phi_0 \subseteq \Phi^*$ s.t. $|\Phi_0| < \kappa \leq 2^\lambda$.

Therefore, there is some $f : \lambda \to 2$ s.t. $\varphi_f \notin \Phi_0$. But now interpret c_α by f, and that will satisfy Φ_0. q.e.d.

The method of ultraproducts

If U is an ultrafilter on κ and let $(M_\alpha\ ;\ \alpha<\kappa)$ be a sequence of first-order structures.

Write $\vec{M}$ for this sequence.

Consider choice functions for $\vec{M}$:

$$f : \kappa \longrightarrow \bigcup_{\alpha<\kappa} M_\alpha$$

s.t. $f(\alpha) \in M_\alpha$

Say $f \sim_U g \iff$

$$\{\alpha\ ;\ f(\alpha) = g(\alpha)\} \in U$$

This is an equivalence relation on the set of choice functions.

$\boxed{\text{ULTRA-PRODUCT}}$ $Ult(\vec{M}, U)$ is the quotient of the set of choice functions modulo $\sim_U$.

We write $[f]$ for $\{g\ ;\ f \sim_U g\}$.

Check that the obvious definitions

[*Example*. If R is a binary relation symbol

$$[f]R[g] \iff \{\alpha<\kappa;\ M_\alpha \models f(\alpha)Rg(\alpha)\} \in U.]$$

are welldefined and give a structure of the right type.

Theorem (Łoś's Theorem)
If φ is a formula in n variables, then

$$\mathrm{Ult}(\vec{M},U) \models \varphi([f_1],\ldots,[f_n])$$

$$\iff$$

$$\{\alpha<\kappa;\ M_\alpha \models \varphi(f_1(\alpha),\ldots,f_n(\alpha))\} \in U.$$

[Essentially: induction on complexity of φ.]

<u>Observation 1</u> If φ is a sentence, Łoś gives us

$$\mathrm{Ult}(\vec{\mathcal{M}}, U) \models \varphi \iff \{\alpha < \kappa ; \mathcal{M}_\alpha \models \varphi\} \in U.$$

So if all $\mathcal{M}_\alpha \models \varphi$, then so is $\mathrm{Ult}(\vec{\mathcal{M}}, U)$.

<u>Observation 2</u> If $\vec{\mathcal{M}}$ is constant, i.e., if there is an $\mathcal{M}$ s.t. $\mathcal{M}_\alpha = \mathcal{M}$ for all α, then we call $\mathrm{Ult}(\vec{\mathcal{M}}, U)$ an <u>ultrapower</u> and write

$$\mathrm{Ult}(\mathcal{M}, U).$$

In this case, we can define the following map

$$\begin{array}{c} m \longmapsto [c_m] \\ \cap \\ M \end{array}$$

constant function
$c_m : \alpha \longmapsto m$

This is an embedding from $\mathcal{M}$ into $\mathrm{Ult}(\mathcal{M}, U)$.

Observation 3 This map

$$j_U : m \longmapsto [c_m]$$

called the ultrapower embedding is an elementary embedding (by Łoś):

[Need to show: $m_1, \dots, m_n \in M$

$$\mathcal{M} \models \varphi(m_1, \dots, m_n) \iff \mathrm{Ult}(\mathcal{M}, U) \models \varphi(j_U(m_1), \dots, j_U(m_n))$$

$$\iff \mathrm{Ult}(\mathcal{M}, U) \models \varphi([c_{m_1}], \dots, [c_{m_n}])$$

$$\overset{\text{Łoś}}{\iff} \{\alpha < \kappa ;\ \mathcal{M} \models \varphi(c_{m_1}(\alpha), \dots, c_{m_n}(\alpha))\} \in U$$

$$\iff \{\alpha < \kappa ;\ \underbrace{\mathcal{M} \models \varphi(m_1, \dots, m_n)}\} \in U$$

This is independent of α, so either true or false.
If true: $\{\dots\} = \kappa \in U$
If false: $\{\dots\} = \emptyset \notin U$.

$$\iff \mathcal{M} \models \varphi(m_1, \dots, m_n)]$$

We can think of $\mathcal{M}$ as an elementary substructure of $\mathrm{Ult}(\mathcal{M}, U)$ by identifying $\mathcal{M}$ with $\{j_U(m) ;\ m \in M\}$.

Theorem (B) measurable $\Longrightarrow$ w.c.

Proof. Suppose κ is measurable. Then we know that κ is inaccessible.

[**Lemma.** If L is an $\mathcal{L}_{\kappa\kappa}$-language with at most κ many non-logical symbols, then $|L| \leq \kappa$.

Proof. Clearly, at most κ many atomic formulas.

Inductively, we only need to show that the construction steps for $\mathcal{L}_{\kappa\kappa}$-languages preserve "card. $\leq \kappa$", i.e., if X is a set of formulas with $|X| \leq \kappa$, then the closure of X under the $\mathcal{L}_{\kappa\kappa}$-constructions has card. $\leq \kappa$.

See also Jech Thm 5.20 p. 57

In this case, closing under $\bigwedge_{\alpha<\lambda} \varphi_\alpha$ will give cardinality $\kappa^{<\kappa} = \bigcup_{\lambda<\kappa} \kappa^\lambda$. Since κ is inaccessible, $\kappa^{<\kappa} = \kappa$.

Similarly, $\exists^\lambda \vec{x}\varphi$ will give rise to $\kappa^{<\kappa}$ many formulas, so $\leq \kappa$.]

If Φ is any set of formulas in an $\mathcal{L}_{\kappa\kappa}$-language with $\leq \kappa$ nonlogical symbols, I can write it as

$$\underline{\Phi = \{\varphi_\alpha ; \alpha < \kappa\}}$$

by our lemma.

Need to show: if Φ is κ-satisfiable, then it's satisfiable.

Define $\Phi_\lambda := \{\varphi_\alpha ; \alpha < \lambda\}$. By assumption, Φ_λ is satisfiable. Let

$M_\lambda \models \Phi_\lambda$.

Define $M := \mathrm{Ult}(\vec{M}, U)$ ← a nontrivial κ-complete uf. on κ [exists by measurability]

<u>Claim</u> $M \models \Phi$.

Let φ_α be arbitrary.

$\{\lambda < \kappa ; M_\lambda \models \varphi_\alpha\}$

$\supseteq \{\lambda < \kappa ; \alpha < \lambda\} \in U$.

So $\{\lambda < \kappa ; M_\lambda \models \varphi_\alpha\} \in U$

$\overset{\text{Łoś}}{\Longleftrightarrow} \mathrm{Ult}(\vec{M}, U) \models \varphi_\alpha$.

q.e.d.

REMINDER

If $|A| < \kappa$, then $A \notin U$ by κ-completeness + non-triviality.

Next goal

inaccessible $\not\Rightarrow$ w.c.

Technique : REFLECTION and the Keisler Extension Property.

If κ is weakly compact, there is a transitive set

$$X \supsetneq V_\kappa$$

s.t.

$$(V_\kappa, \in) \preccurlyeq (X, \in).$$

[Compare the situation of worldly cardinals.]