LARGE CARDINALS

VII

14 February 2022

GOAL

STRONGLY COMPACT ?

$\Downarrow$

MEASURABLE

$\Downarrow$

WEAKLY COMPACT ?

$\Downarrow$

INACCESSIBLE

First : Implications.

Later: Non-implications

showing that these notions are ordered in LOGICAL STRENGTH

[To be discussed in detail later.]

We shall consider infinitary languages which are generalizations of the ordinary first order language. Let κ be an infinite cardinal number. The language $\mathcal{L}_{\kappa,\omega}$ consists of

(i) κ variables;
(ii) various relation, function, and constant symbols;
(iii) logical connectives and infinitary connectives $\bigvee_{\xi<\alpha}\varphi_\xi$, $\bigwedge_{\xi<\alpha}\varphi_\xi$ for $\alpha<\kappa$ (infinite disjunction and conjunction);
(iv) quantifiers $\exists v$, $\forall v$.

NON-LOGICAL SYMBOLS
S

The language $\mathcal{L}_{\kappa,\kappa}$ is like $\mathcal{L}_{\kappa,\omega}$ except that it also contains infinitary quantifiers:

(v) $\exists_{\xi<\alpha}v_\xi$, $\forall_{\xi<\alpha}v_\xi$ for $\alpha<\kappa$.

The interpretation of the infinitary symbols of $\mathcal{L}_{\kappa,\kappa}$ is the obvious generalization of the finitary case where $\bigvee_{\xi<n}\varphi_\xi$ is $\varphi_0\vee\ldots\vee\varphi_{n-1}$, $\exists_{\xi<n}v_\xi$ stands for $\exists v_0\ldots\exists v_{n-1}$, etc. The language $\mathcal{L}_{\omega,\omega}$ is just the language of the first order predicate calculus.

$\mathcal{L}_{\kappa\lambda}$ - languages

κ: size of conjunctions & disjunction
λ: size of quantifiers

$\bigwedge_{\alpha<\mu}\varphi_\alpha$ where $\mu<\kappa$

$\bigvee_{\alpha<\mu}\varphi_\alpha$

$\exists^\nu \vec{x}\,\varphi$
$\forall^\nu \vec{x}\,\varphi$
where $\vec{x}$ is a function from ν into the set of variables

An $\mathcal{L}_{\kappa\lambda}$ - language is constructed as follows:

1. Terms are constructed from S exactly as in first order logic.
2. Atomic formulas are defined exactly as in first order logic.

3. $\mathfrak{M} \models \bigwedge_{\alpha<\mu} \varphi_\alpha \quad :\Longleftrightarrow$

for all $\alpha<\mu$ $\mathfrak{M} \models \varphi_\alpha$ ..

Similarly for $\bigvee_{\alpha<\mu} \varphi_\alpha$.

4. $\mathfrak{M} \models \exists^\gamma \vec{x}\, \varphi \quad :\Longleftrightarrow$

there is function $a : \gamma \longrightarrow M$

s.t.

$$\mathfrak{M}\left[\vec{x}(\alpha) \longmapsto a(\alpha)\right] \models \varphi$$

the variable $\vec{x}(\alpha)$
is interpreted by
$a(\alpha)$

Infinitary logic can express multiple features not expressible in FOL:

(i) $\quad \forall^\omega \vec{x} \bigvee_{i \neq j} x_i = x_j = \varphi_{fin}$

Clearly $\mathfrak{M} \models \varphi_{fin}$ iff $\mathfrak{M}$ is a finite structure

φ_{fin} is an $\mathcal{L}_{\omega_1\omega_1}$-formula.

<u>Remark</u> This immediately implies that the usual formulation of compactness for FOL cannot be true here.

(ii) If c_n is a constant symbol for each $n \in \mathbb{N}$, then

$$\forall x \bigvee_{n \in \mathbb{N}} x = c_n$$

describes model that are at most countable.

Furthermore, if we add

$$\bigwedge_{n \neq m} c_n \neq c_m,$$

then the model is in bijection with $\mathbb{N}$.

<u>Def.</u> If Φ is a set of formulas in an $\mathcal{L}_{\kappa\lambda}$-language, we say that Φ is <u>μ-satisfiable</u> if for each subset $\Phi_0 \subseteq \Phi$ s.t. $|\Phi_0| < \mu$, Φ_0 is satisfiable.

So: finitely satisfiable $\Longleftrightarrow$ $\aleph_0$-satisfiable.

Def. A cardinal κ is called strongly-compact if for every $\mathcal{L}_{\kappa\kappa}$-language

s.c. L and every Φ of L-sentences:

if Φ is κ-satisfiable,
then it is satisfiable

A cardinal κ is called weakly compact if for every $\mathcal{L}_{\kappa\kappa}$-language L with

w.c. set of nonlogical symbols $|S| \leq \kappa$

and every Φ of L-sentences:

if Φ is κ-satisfiable,
then it is satisfiable.

WARM-UP

Theorem 1 If κ is w.c., then κ is regular.

Proof. Suppose $\kappa = \bigcup X$ where $|X| < \kappa$.

Take an $\mathcal{L}_{\kappa\kappa}$-language with constant symbols c_α $(\alpha < \kappa)$ and c. κ many nonlogical symbols

$c_\alpha \neq c$ [for $\alpha < \kappa$]

$$\underbrace{\bigvee_{\beta\in X}}_{\text{disjunction of size } |X|<\kappa}\ \underbrace{\bigvee_{\alpha<\beta}}_{\text{disjunction of size } \beta<\kappa} c=c_\alpha \quad =: \psi$$

Let $\Phi := \{\psi\} \cup \{c_\alpha \neq c ;\ \alpha<\kappa\}$

Clearly, Φ is κ-satisfiable, but not satisfiable. Contradiction! q.e.d.

<u>Theorem 2</u> (Keisler–Tarski)

If κ is s.c. and F is a κ-complete filter on an arbitrary set X, then F can be extended to a κ-complete ultrafilter on X.

<u>Remark.</u> Using AC, every filter can be extended to an ultrafilter. The strength of this theorem is that we're preserving κ-completeness!

Proof. Fix X and F κ-complete filter on X.
w.l.o.g. X is an ordinal (by AC).

If $A \subseteq X$, let c_A be a constant symbol.
Add an additional constant symbol c.

In total, $2^{|X|}$ many nonlogical symbols.

Let L be the $\mathcal{L}_{\kappa\kappa}$-language with $\in$ and nonlogical symbols c_A;
let L^* be the $\mathcal{L}_{\kappa\kappa}$-language with $\in, c_A, c$.

$$\mathcal{M} := (\wp(X), \in, \{A; A \subseteq X\})$$

and $\Phi := \mathrm{Th}_L(\mathcal{M})$.

Note α is an ordinal $\iff$ $\mathcal{M} \models \alpha$ is a transitive set totally ordered by $\in$

If $A \subseteq X$, we have

$\mathcal{M} \models \forall x\,(x \text{ is an ordinal} \longrightarrow (x \in c_A \vee x \in c_{X \setminus A}))$

$\mathcal{M} \models \forall x\,(x \in c_A \longrightarrow x \text{ is an ordinal})$

Now let $\Phi^* := \Phi \cup \{c \in c_A \,;\, A \in F\}$.

Then Φ^* is κ-satisfiable,

[if $\Phi_0 \subseteq \Phi^*$ has size $< \kappa$, then the intersection of all A s.t. "$c \in c_A$" occurs in Φ_0 is non-empty, so any element of the intersection can serve as interpretation of c]

thus Φ^* is satisfiable.

Let $\mathcal{M}^*$ be a model of Φ^* and define

$$U := \{ A \subseteq X ; \mathcal{M}^* \models c \in c_A \} .$$

By the above, $\mathcal{M}^* \models c$ is an ordinal.

We need to show that U is a κ-complete uf. on κ.

1. Since $c \in c_A$ for $A \in F$ is in Φ^*, we immediately get that $F \subseteq U$.
2. Since $\mathcal{M} \models \forall x (x \text{ ordinal} \longrightarrow (x \in c_A \vee x \in c_{X \setminus A}))$, and $\mathcal{M}^* \models c$ is an ordinal, we have that U is an ultrafilter.
3. Let's show κ-completeness. Let A_α $(\alpha < \lambda)$ be s.t.

$$\boxed{\mathcal{M}^* \models c \in c_{A_\alpha}}$$

Define $A := \bigcap_{\alpha < \lambda} A_\alpha$ (*)

Want to see that $\mathcal{M}^* \models c \in c_A$.

(*) can be expressed in $\mathcal{L}_{\kappa\kappa}$ as follows

$$\mathcal{M} \models \forall x . (x \in c_A \longleftrightarrow \bigwedge_{\alpha < \lambda} x \in c_{A_\alpha})$$

$$\Longrightarrow \mathcal{M}^* \models c \in c_A \longleftrightarrow \bigwedge_{\alpha < \lambda} c \in c_{A_\alpha}$$

By assumption, RHS is true, so

$$\mathcal{M}^* \models c \in C_A$$

$$\Longrightarrow \quad A \in U.$$

q.e.d.

<u>Corollary</u> Any s.c. cardinal is measurable.

<u>Proof.</u> [measurable: nontrivial κ-complete ultrafilter on κ]

To show: existence of a κ-complete filter on κ s.t. no extension to uf. can be trivial.

$$\mathcal{F} := \{A \subseteq \kappa ; |\kappa \setminus A| < \kappa\}$$

By our warm-up theorem, κ is regular and so $\mathcal{F}$ is κ-complete.

Therefore by Keisler-Tarski Thm, it can be extended to a κ-complete ultrafilter.

Clearly, that cannot be trivial. q.e.d.