LARGE CARDINALS
Lecture VI

9 Feb 2022

LARGER LARGE CARDINALS

- Strongly compact cardinals (s.c.)
- Measurable cardinals
- Weakly compact cardinals (w.c.)
- Inaccessible cardinals

GOAL: Show implications.

Every s.c. is measurable.
Every measurable is w.c.
Every w.c. is inaccessible.

After that: non-implications.

Henri Lebesgue

Born	June 28, 1875 Beauvais, Oise, France
Died	July 26, 1941 (aged 66) Paris, France
Nationality	French

Giuseppe Vitali

Born	26 August 1875 Ravenna, Italy
Died	29 February 1932 (aged 56) Bologna, Italy

Measure problem

(for the unit interval $[0,1]$)

Is there a function

$$\mu : \mathcal{P}([0,1]) \longrightarrow [0,1]$$

s.t.

- $\mu([0,1]) = 1$
- $\mu(\emptyset) = 0$
- translation invariance
  $$\mu(A+x) = \mu(A)$$
  where $+x$ refers to a translation
- σ-additivity
  if $\{A_i \,;\, i \in \mathbb{N}\}$ is a family of p.w. disjoint sets, ~~then~~
  $$\mu\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = \sum_{i\in\mathbb{N}} \mu(A_i)$$

?

Answer (Vitali 1905): No.

More precisely: AC $\Longrightarrow$ there is a non-measurable set.

Even more precisely: if there is a basis for $\mathbb{R}$ as $\mathbb{Q}$-vector space, then answer is No.

Answer in measure theory:
Give up on $\mathrm{dom}(\mu) = \mathcal{P}([0,1])$
and work with appropriate σ-algebras instead.

First generalisation:
<u>Banach</u> DROP translation invariance.
If you do so, you could have trivial solutions:
if $\mu(A) = 1$ if $0 \in A$
$\mu(A) = 0$ if $0 \notin A$
This is a solution to the measure problem without translation invariance.

<u>Banach</u> μ is <u>nontrivial</u> if for all x
$\mu(\{x\}) = 0$.
Therefore (by σ-additivity), all countable sets get measure 0.

One further abstraction step:

Stanisław Ulam

Stanisław Ulam

Born	Stanisław Marcin Ulam 13 April 1909 Lemberg, Austria-Hungary (now Lviv, Ukraine)
Died	13 May 1984 (aged 75) Santa Fe, New Mexico, U.S.
Nationality	Polish
Citizenship	Poland, United States (naturalized in 1941)
Education	Lwów Polytechnic Institute, Second Polish Republic

<u>Def.</u> An <u>ULAM measure on X</u> is a function

$$\mu : \mathcal{P}(X) \longrightarrow \{0,1\}$$

that is nontrivial, $\mu(X)=1$, $\mu(\emptyset)=0$, and μ is <u>σ-additive.</u>

[Note: if $A \cap B = \emptyset$, then at most one of A and B can have measure 1.]

<u>Definition</u> A cardinal κ is called <u>ULAM MEASURABLE</u> if there is an Ulam measure on κ.

(Possibly something on ES #2.)

If μ is an Ulam measure on κ, we call it <u>κ-additive</u> if for all families $\{A_\alpha ; \alpha < \lambda\}$ for $\lambda < \kappa$ s.t. the A_α's are p.w. disjoint

$$\mu\Big(\bigcup_{\alpha<\lambda} A_\alpha\Big) = \underbrace{\sum_{\alpha<\lambda} \mu(A_\alpha)}$$

$= 1$ iff there is $\alpha<\lambda$ $\mu(A_\alpha)=1$

Def. A cardinal κ is *measurable* if there is a κ-additive Ulam measure on it.

[If $A \subseteq \kappa$ has card. $|A| < \kappa$, then $\mu(A) = 0$.]

Filters & Ultrafilters.

A family $F \subseteq \wp(\kappa)$ is called a *filter on κ* if

$$\kappa \in F$$

$$\emptyset \notin F$$

"$\aleph_0$-complete"

$$A, B \in F \longrightarrow A \cap B \in F$$

$$A \in F,\ B \supseteq A \longrightarrow B \in F$$

A filter is *λ-complete* if for all $\mu < \lambda$ and $\{A_\alpha ; \alpha < \mu\} \subseteq F$, we have

$$\bigcap_{\alpha < \mu} A_\alpha \in F.$$

A filter is *nontrivial* if for all $\alpha \in \kappa$,

$$\{\alpha\} \notin F$$

A filter is called *ultrafilter* if for all $A \subseteq \kappa$, either $A \in F$ or $\kappa \setminus A \in F$.

OBSERVATION

κ-complete non-trivial ultrafilters and κ-additive Ulam measures are the same thing.

[U ultrafilter $\rightsquigarrow$ $\mu(A)=1 :\Leftrightarrow A\in U$

μ measure $\rightsquigarrow$ $A\in U :\Leftrightarrow \mu(A)=1$.

Need to check via set algebra that additivity & completeness translate into each other in this particular case.

E.g., if U is ultrafilter, κ-completeness is eq. to

(*) "if $\{A_\alpha ; \alpha<\lambda\}$ s.t. $A_\alpha \notin U$, then $\bigcup_{\alpha<\lambda} A_\alpha \notin U$.]

Theorem 1 If κ is measurable, κ is regular.

Proof. Let U be the κ-complete uf. on κ. By (*), sets of size $<\kappa$ cannot be in U. If $\kappa = \bigcup C$ where C is cofinal and $|C|<\kappa$,

then each $\alpha \in C$ has size $< \kappa$,
so $\alpha \notin U$, but then κ is a
$|C|$-union of things not in U,
so by κ-completeness $\kappa \notin U$.
Contradiction! q.e.d.

<u>Theorem 2</u> If κ is measurable, then
κ is a ~~strong~~ limit.

<u>Corollary</u> All measurables are inaccessible.

<u>Pf of Thm 2</u> If κ is not a ~~strong~~ limit,
then there is $\lambda < \kappa$ s.t. $2^\lambda \geq \kappa$.

This means there is an injection
$$f : \kappa \longrightarrow \{F ; F : \lambda \to 2\}$$
Write $f_\alpha : \lambda \to 2$ for $f(\alpha)$.

Fix $\boxed{\xi \in \lambda}$, consider
$$\kappa = \{\alpha ; f_\alpha(\xi) = 0\} \cup \{\alpha ; f_\alpha(\xi) = 1\}$$
Precisely one of these sets is in U. Let b_β
be either $0, 1$ depending on which set is in U.
$$U \ni A_\xi := \{\alpha ; f_\alpha(\xi) = b_\xi\}$$

Since $A_\xi \in U$ for each $\xi < \lambda$, and U is κ-complete, have

$$A := \bigcap_{\xi<\lambda} A_\xi \in U.$$

$$= \bigcap_{\xi<\lambda} \{\alpha ; f_\alpha(\xi) = b_\xi\}$$

$$= \{\alpha ; \forall \xi < \lambda \ f_\alpha(\xi) = b_\xi\}$$

But $|A| \leq 1$, so $A \notin U$.

Contradiction! q.e.d.

MEASURABLE CARDINALS will feature prominently in the second half of this course.

For now, all we know is that they <u>are</u> large cardinals.

COMPACTNESS

In FOL, compactness is the statement
if Φ is finitely satisfiable (every finite subset is satisfiable)

then Φ is satisfiable.

Compactness is very powerful but also the source of limitations of FOL:

E.g., cannot characterise finite structures, countable structures etc.

Tarski's idea Expand FOL to allow infinite formulas. INFINITARY LOGICS

Let κ and λ be cardinals. We define $\mathcal{L}_{\kappa\lambda}$ languages by:

1. allowing arbitrary families of variables
2. S nonlogical symbol [could be infinite / uncountable]
3. $\wedge, \vee, \neg, \Rightarrow, \exists, \forall$ as usual
4. $\bigwedge_{\alpha<\mu} \varphi_\alpha$ where $\mu < \kappa$

   $\bigvee_{\alpha<\mu} \varphi_\alpha$

5. $\exists^\mu \vec{x} \varphi$, $\forall^\mu \vec{x} \varphi$ where $\mu < \lambda$

$\vec{x}$ is a seq. of var. of length μ