Large Cardinals

Fifth Lecture

7 February 2022

Comment on weakly inaccessibles.

Goal: $ZFC \nvdash WIC$

where WIC is "there is a weakly inaccessible".

$M \subseteq N \quad (M, \in) \models ZFC$

$(N, \in) \models ZFC$

We defined "M is transitive in N".

Example: $V_\lambda \subseteq V_\kappa \Longrightarrow V_\lambda$ is transitive in V_κ.

E.g., κ is inaccessible & λ is the constructed worldly card. below κ, then $(V_\lambda, \in) \models ZFC$ and $(V_\kappa, \in) \models ZFC$.

Q. (J.B.) Can we give an example where M is not transitive in N.

We saw: Δ_0 formulas are absolute between transitive models.

<u>Trivial</u> example of non-preservation of Δ_0 formulas for non-transitive models:

Let $(N, \in) \models ZFC$ and

$M := \{0, 2\}$

not a model of ZFC

$\varphi(x)$ = there is exactly one element of x

$$= \exists u\,(u \in x \wedge u = u) \wedge \forall v\,(v \in x \rightarrow \forall w\,(w \in x \rightarrow v = w))$$

This is Δ_0.

$(M, \in) \models \varphi(2)$ but

$(N, \in) \not\models \varphi(2)$.

Later, we'll see a non-transitive $M \subseteq N$ s.t. $(M, \in) \models ZFC$.

Def. A formula is called Σ_1 / Π_1 if it is of the form

$$\exists x\, \psi \quad \text{or}$$

$$\forall x\, \psi$$

where ψ is a Δ_0 formula.

Def. A formula φ is called downwards / upwards absolute between M and N if for all $\vec{x} \in M^n$ we have

$$N \models \varphi(\vec{x}) \underset{\Longleftarrow}{\Longrightarrow} M \models \varphi(\vec{x})$$

Proposition Σ_1 formulas are upwards absolute and Π_1 formulas are downwards absolute for transitive models $\boxed{\text{of ZFC}}$.

Proof. This follows directly from the fact that Δ_0 formulas are absolute and $M \subseteq N$:

$$(M, \in) \models \exists x\, \psi \Longrightarrow \text{there is } a \in M \; (M,\in) \models \psi(a)$$

$$\Longrightarrow \text{there is } a \in N \; (M,\in) \models \psi(a)$$

$$\Longrightarrow (N, \in) \models \exists x\, \psi. \qquad \text{q.e.d.}$$

Examples

The following properties are described in ZFC by Π_1-formulas:

1. κ is a cardinal
2. κ is a regular cardinal
3. κ is a limit cardinal
4. κ is a strong limit cardinal

CONSEQUENCE

If $M \subseteq N$ is transitive model of ZFC and $\kappa \in M$ and

$N \models \kappa$ is (weakly) inaccessible

$\Longrightarrow M \models \kappa$ is (weakly) inaccessible.

GCH GENERALISED CONTINUUM HYPOTHESIS

for all α $2^{\aleph_\alpha} = \aleph_{\alpha+1}$

Note! κ is a limit cardinal $\iff \forall \alpha\, \aleph_\alpha < \kappa \rightarrow \aleph_{\alpha+1} < \kappa$

κ is a strong limit card $\iff \forall \alpha\, \aleph_\alpha < \kappa \rightarrow 2^{\aleph_\alpha} < \kappa$

So GCH $\Rightarrow$ limit = strong limit

Thus:

$$\text{GCH} \Rightarrow \text{weakly inacc.} = \text{strongly inacc.}$$

Let WIC $:= \exists \kappa\,(\kappa$ is weakly inaccessible$)$

We're going to argue: if ZFC is consistent, then ZFC $\nvdash$ WIC.

<u>Def.</u> $M \subseteq N$ M transitive in N, both models of ZFC

We say M is an inner model of N if $\mathrm{Ord} \cap M = \mathrm{Ord} \cap N$.

M is definable in N if there is a formula Φ s.t.

$$x \in M \Longleftrightarrow N \models \Phi(x).$$

<u>Proposition</u> If M is definable in N, then the relation $M \models \varphi$ is first-order expressible in N.

<u>Proof.</u> The method of RELATIVISATION

Translate φ into a relativised φ^M s.t.

$$N \models \varphi^M \Longleftrightarrow M \models \varphi.$$

Do this by recursion on the formulas:

$$\varphi^M := \varphi \quad \text{if } \varphi \text{ is atomic}$$

$$(\varphi \wedge \psi)^M := \varphi^M \wedge \psi^M$$

$$(\neg\varphi)^M := \neg(\varphi^M)$$

$$(\exists x \varphi)^M := \exists x (\Phi(x) \wedge \varphi)$$

By an easy induction using the def'n, we get

$$M \models \varphi \iff N \models \varphi^M \qquad \text{q.e.d.}$$

<u>Theorem</u> (Gödel 1938)

If $(N, \in) \models ZFC$, then there is a definable inner model M s.t. $(M, \in) \models ZFC + GCH$.

<u>Remarks</u> ① This implies $\mathrm{Cons}(ZFC) \Rightarrow \mathrm{Cons}(ZFC + GCH)$.

② There is a uniform formula Φ defining M in every model of ZFC

L Gödel's Constructible Universe

③ You do not need AC in N, so the theorem also shows

$$\mathrm{Cons}(ZF) \Longrightarrow \mathrm{Cons}(ZFC).$$

Corollary If ZFC is consistent, then $ZFC \nvdash WIC$.

Proof. Let $(N, \in) \models ZFC$. By Gödel 1938 there is a definable $M \subseteq N$ s.t.

$$(M, \in) \models ZFC + GCH.$$

Towards a contradiction, assume $ZFC \vdash WIC$, so

$$(M, \in) \models ZFC + GCH + WIC$$

$$\Longrightarrow (M, \in) \models ZFC + IC.$$

Redo Hausdorff's proof to show that if κ is an inaccessible card. in M, then

$$V_\kappa^M := \{ x \in M ; \boxed{M \models \varrho(x) < \kappa} \}$$

is a set in N by the fact that ↑ can be expressed in a first-order formula.

Hausdorff's proof gives that

$$N \models \text{"} V_\kappa^M \models ZFC \text{"}.$$

So we argued that $ZFC \vdash \mathrm{Cons}(ZFC)$. q.e.d.

Contradiction to Incompleteness.

DIGRESSION

A non-transitive $M \subseteq N$ s.t.

$$M \models ZFC$$

This is an improvement of ↓LöSko to obtain a ctble elementary substructure of N.

The proof technique (using TVT) is precisely the one used in our proof of the existence of worldly cardinals.

Idea Do the same proof but only include the witnesses:

$$M_0 := \emptyset$$

If M_i is defined (and countable) then consider all $\vec{x} \in M_i^{<\omega}$ and formulas φ and consider

$$N \models \exists x\, \varphi(x, \vec{x})$$

if so, let $w(\varphi, \vec{x})$ be a witness

if not, let $w(\varphi, \vec{x}) := \emptyset$

$$M_{i+1} := M_i \cup \{ w(\varphi, \vec{x}) ;\ \underset{\vec{x} \in M_i^{<\omega}}{\varphi \text{ fmla}} \}$$

By induction, M_i is countable and so is $M := \bigcup_{i \in \mathbb{N}} M_i$

[as a ctble union of ctble sets]

The very same TVT argument as before shows

$$M \preccurlyeq N.$$

In particular $M \models ZFC$. If $N = V_\kappa$ where κ is inaccessible or worldly, then $M \neq N$.

<u>Claim</u> M cannot be transitive in N.

Consider the formula $\varphi(x) =$ "x is the smallest uncountable ordinal".

$$N \models \exists x\, \varphi(x)$$

This formula has precise one witness in N, viz. $x = \aleph_1$.

So, $\aleph_1 \in M_1 \subseteq M$. By countability of M, we have $\aleph_1 \cap M$ is countable, so $\aleph_1 \not\subseteq M$. So M is not transitive.