LARGE CARDINALS

IX

21 FEBRUARY 2022

$$\begin{array}{ll} \text{STRONGLY COMPACT} & SC \\ \Downarrow & \\ \text{MEASURABLE} & MC \\ \Downarrow & \\ \text{WEAKLY COMPACT} & WC \\ \Downarrow \ \not\Uparrow \ \text{REFLECTION} & \\ \text{INACCESSIBLE} & IC \end{array}$$

KEISLER EXTENSION PROPERTY

κ has the Keisler Extension Property (KEP) if there is a transitive set $X \supsetneq V_\kappa$ s.t.

$$(V_\kappa, \in) \preccurlyeq (X, \in).$$

Theorem If κ is weakly compact, then κ has the KEP.

Proof. Observe that κ is inaccessible, this means (by Hausdorff):

- if $x \in V_\kappa$, then $|x| < \kappa$
- $|V_\kappa| = \kappa$

We define an $\mathcal{L}_{\kappa\kappa}$ language with additional constant symbols:

L	c_x	for each $x \in V_\kappa$
L^*	c	in addition

[Note that this language has κ many non-logical symbols.]

$$\mathcal{V} := (V_\kappa, \in, \{x ; x \in V_\kappa\})$$

is an L-structure.

Let

$$\Phi := \mathrm{Th}_L(\mathcal{V}).$$

$$\boxed{\Phi^* := \Phi \cup \{c \neq c_x ; x \in V_\kappa\} \cup \{c \text{ is an ordinal}\}}$$

L is the $\mathcal{L}_{\kappa\kappa}$ language with $\{c_x ; x \in V_\kappa\}$

L^* is the $\mathcal{L}_{\kappa\kappa}$ lang. with $\{c_x ; x \in V_\kappa\} \cup \{c\}$.

Observe that the formula

$$\forall^{\omega} \vec{x} \bigvee_{i\in\mathbb{N}} x_{i+1} \notin x_i =: \psi.$$

describes wellfoundedness and is an $\mathcal{L}_{\omega_1\omega_1}$-formula.

Clearly $\gamma \models \psi$, so $\psi \in \Phi$, and so any model of $\Phi^* \supseteq \Phi$ will be $\boxed{\text{wellfounded.}}$

Also, since κ is inaccessible

$$ZFC \subseteq \Phi,$$

in particular, the axiom of extensionality, and therefore any model of $\Phi^* \supseteq \Phi$ will be $\boxed{\text{extensional.}}$

Clearly, Φ^* is κ-satisfiable.

[If $\Phi_0 \subseteq \Phi^*$ with $|\Phi_0| < \kappa$, then there is an ordinal $\alpha < \kappa$ s.t. $c \neq c_\alpha \notin \Phi_0$. Interpret c by α.]

By weak compactness, Φ^* is satisfiable. Let $M \models \Phi^*$.

We have $\mathcal{M} \models \Phi^*$.
By the above, $\mathcal{M}$ is wellfounded and extensional.

LEADER, Logic & Set Theory NOTES, §5

The analogue of 'subset collapse' is:

Theorem 4 (Mostowski's Collapsing Theorem). Let r be a relation on a set a that is well-founded and extensional. Then there exists a transitive set b, and a bijection $f : a \to b$ such that $(\forall x, y \in a)(x \; r \; y \Leftrightarrow f(x) \in f(y))$. Moreover, b and f are unique.

Remark. 'Well-founded' and 'extensional' are trivially necessary.

So, by Mostowski, there is a transitive X s.t.

Write $\mathfrak{X} := (X, \in, \dots)$

$$(X, \in) \cong (M, E).$$

First show that $V_\kappa \subseteq X$:

$$\gamma \models \forall z \Big(z \in c_x \longleftrightarrow \bigvee_{y \in x} z = c_y \Big)$$

uses that $|x| < \kappa$

So, this formula is in Φ, thus true in $\mathcal{M}$ and in $(X, \in)$.
Thus by a simple induction on rank, we get that $\boxed{c_x^{\mathfrak{X}} = x.}$

Since $\mathfrak{X} \models \Phi^*$, we get that

$c^{\mathfrak{X}}$ is an ordinal

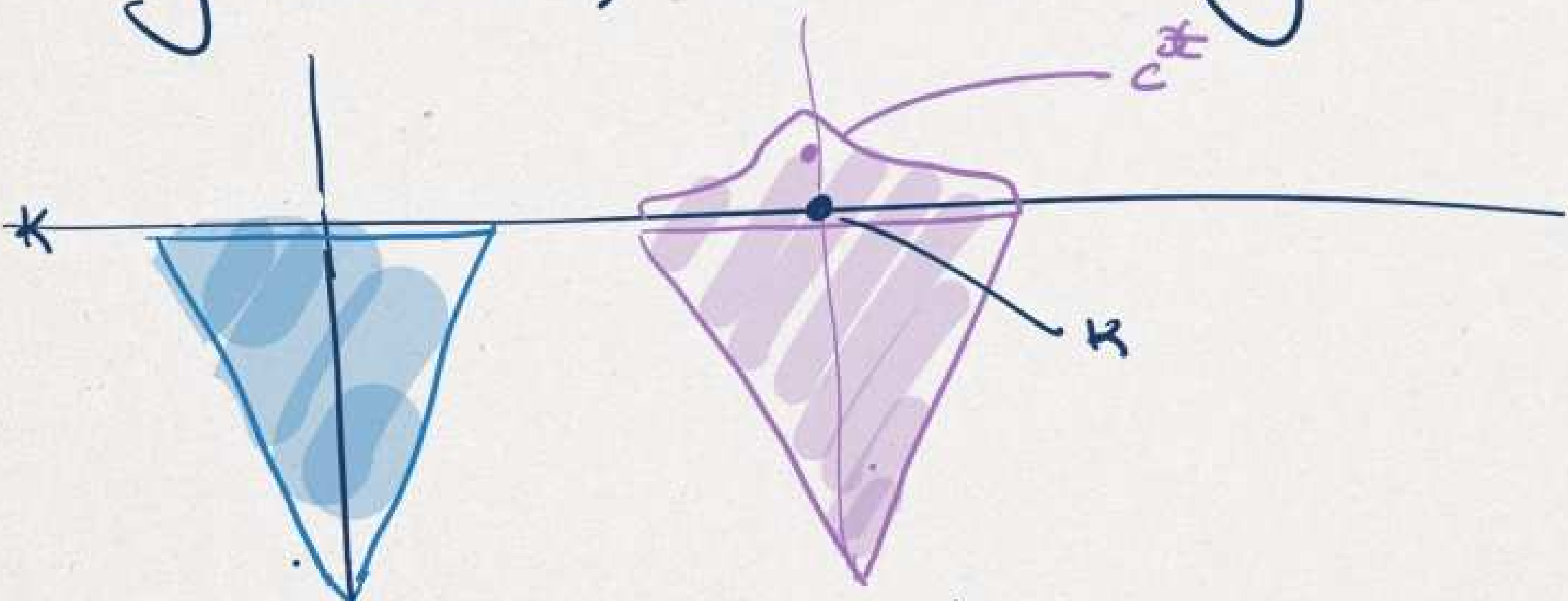
and $c^{\mathfrak{X}}$ is bigger than all $\dot{\iota} < \kappa$.

Thus $X \supsetneqq V_\kappa$.

Therefore $x \longmapsto c_x^{\mathfrak{X}} = x$

is the identity.

By construction, this is elementary.



Note that $\kappa \in X$.

<u>Theorem</u> If κ is weakly compact, then there is $\lambda < \kappa$ inaccessible.

<u>Corollary</u> The least inaccessible cannot be weakly compact.

Proof of Theorem

By KEP, find $X \supsetneq V_\kappa$ s.t.

$$(V_\kappa, \in) \preccurlyeq (X, \in).$$

As we saw in the proof, $\kappa \in X$.

STEP 2. Inaccessibility is downwards absolute for transitive sets, so

$$\mathfrak{X} \models \kappa \text{ is inaccessible}.$$

So

$$\mathfrak{X} \models \text{IC}.$$

However, $(V_\kappa, \in) \equiv (X, \in)$, so

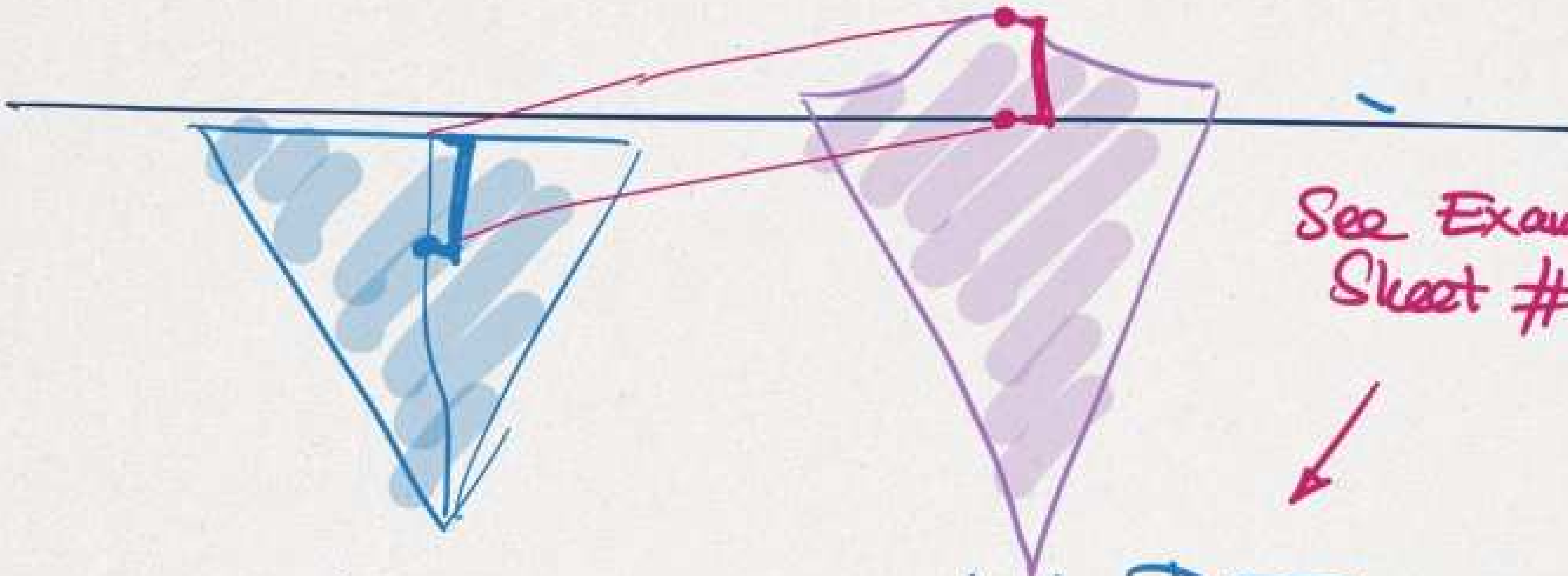
$$V_\kappa \models \text{IC}.$$

So there is $\lambda < \kappa$ s.t.

$$V_\kappa \models \lambda \text{ is inaccessible}.$$

STEP 5. Thus (by ES#2), λ is inaccessible.

q.e.d.



See Example Sheet #2.

This phenomenon is called **REFLECTION**:
A property that holds in the taller universe $\mathfrak{X}$ is reflected downwards in the shorter universe.

A step-by-step analysis of this argument:

1. Suppose $\Phi(\kappa)$ holds.
2. Argue that $\mathfrak{X} \models \Phi(\kappa)$.
3. Therefore $\mathfrak{X} \models \exists \alpha\, \Phi(\alpha)$
4. Use elementarity to get
   $\mathfrak{Y} \models \exists \alpha\, \Phi(\alpha)$.
5. Argue that there is $\alpha < \kappa$ s.t. $\Phi(\alpha)$.

LOGICAL HIERARCHIES

How do we measure the logical strength of theories / large cardinal axioms?

First idea "Logical consequence".

If Φ is a set of sentences, then define $C_\Phi := \{\psi ; \Phi \vdash \psi\}$.

Let $\Phi \leq_0 \Psi :\Longleftrightarrow C_\Phi \subseteq C_\Psi$.

In particular, we have

$$\mathrm{ZFC} \leq_0 \mathrm{ZFC}+\mathrm{IC} \leq_0 \mathrm{ZFC}+\mathrm{WC} \leq_0 \mathrm{ZFC}+\mathrm{MC} \leq_0 \mathrm{ZFC}+\mathrm{SC}.$$

[In a very strong sense since the properties are implied downwards.]

Also $\Phi \equiv_0 \Psi :\Longleftrightarrow \Phi \leq_0 \Psi \wedge \Psi \leq_0 \Phi$

[i.e., $C_\Phi = C_\Psi$]

$\Phi <_0 \Psi :\Longleftrightarrow \Phi \leq_0 \Psi$ and $\Phi \not\equiv_0 \Psi$.

So, we have now

$$\mathrm{ZFC} <_0 \mathrm{ZFC} + \mathrm{IC}$$

$$\mathrm{ZFC} + \mathrm{IC} <_0 \mathrm{ZFC} + \mathrm{WC}.$$

This hierarchy inherits the complexity from the fact that it is just subset relation.

If Φ is incomplete, then there is φ s.t.

$$\Phi \nvdash \varphi \text{ and } \Phi \nvdash \neg\varphi.$$

Thus $\Phi \cup \{\varphi\}$ and $\Phi \cup \{\neg\varphi\}$ are incomparable in $\leq_0$.

In particular, we cannot compare

$$\mathrm{ZFC} + \mathrm{CH} \text{ and } \mathrm{ZFC} + \neg\mathrm{CH}.$$

This casts some doubt on $\leq_0$ as a good hierarchy.

Second idea

Let Φ be a cardinal property

[i.e., if $\Phi(x)$, then x is a cardinal]

and write

$$\Phi C := \exists x\, \Phi(x)$$

Try to formalise the idea of

"Φ-cardinals are bigger than Ψ-cardinals".

Not possible: "every Φ-cardinal is bigger than every Ψ-cardinal".

Let κ_Φ be the smallest Φ-cardinal (if it exists) and define

$$\Phi \leq_1 \Psi \;:\Longleftrightarrow\; \kappa_\Phi \leq \kappa_\Psi .$$

Similarly, $<_1$, $\equiv_1$.

We proved today, that $IC <_1 WC$.

More discussion of $\leq_1$ in lecture X.