# Large Cardinals
## Lecture IV

2 February 2022

Lecture III: If $\kappa$ inaccessible, then
$$W := \{\lambda \in \kappa \,;\, V_\lambda \preccurlyeq V_\kappa\}$$ and
$$W^{\aleph_0} := \{\lambda \in \kappa \,;\, V_\lambda \preccurlyeq V_\kappa \ \&\ \underline{\mathrm{cf}(\lambda) = \aleph_0}\}$$
have size $\kappa$.

Question: Are there worldly cardinals $\lambda < \kappa$ s.t. $\mathrm{cf}(\lambda) > \aleph_0$?

Two conditions to be a large cardinal:

(a) very big

(b) existence is not provable

IN PERSON OFFICE HOUR!
Friday 4 Feb 2022
11–12
MR14

# Basic model theory

① • $M_0 \preccurlyeq N$
• $M_1 \preccurlyeq N$
$M_0 \subseteq M_1$ $\Bigg\} \Longrightarrow M_0 \preccurlyeq M_1$

[Follows from the definitions.]

In light of lecture III, this means if $\kappa$ is inaccessible, we have lots of pairs $(\lambda, \lambda')$ s.t. $\lambda < \lambda' < \kappa$ with

$$V_\lambda \preccurlyeq V_{\lambda'}.$$

② TARSKI'S CHAIN LEMMA

If $(L, <)$ is a total order and for each $\ell \in L$, $M_\ell$ is some structure, we call $(M_\ell; \ell \in L)$ an elementary chain if for all $\ell < \ell'$, $M_\ell \preccurlyeq M_{\ell'}$.

If $(M_\ell; \ell \in L)$ is an elementary chain and $M := \bigcup_{\ell \in L} M_\ell$, then for each $\ell \in L$, $M_\ell \preccurlyeq M$.

[Proof is a simple adaptation of the proof of the TVT; ES#1.]

<u>Theorem</u> If $\kappa$ is inaccessible and $\mu < \kappa$ is regular, then there is $\lambda < \kappa$ s.t. $\lambda$ is worldly and

$$\mathrm{cf}(\lambda) = \mu.$$

[We cannot in general guarantee that for some $\mu$ we have $\mu = \lambda$.]

<u>Proof.</u>

$$W := \{\lambda \in \kappa \,;\, V_\lambda \preccurlyeq V_\kappa\}$$

has size $\kappa$, so take its increasing enumeration:

$\lambda_\alpha$ is the $\alpha$th element of $W$.

By ① from the last page, if $\alpha < \beta$, then $V_{\lambda_\alpha} \preccurlyeq V_{\lambda_\beta}$.

So, if $X \subseteq \kappa$ is arbitrary

$$\{V_{\lambda_\alpha} \,;\, \alpha \in X\}$$

forms an elementary chain.

So, consider $\{V_{\lambda_\alpha} \,;\, \alpha < \mu\}$. Then $\bigcup_{\alpha<\mu} V_{\lambda_\alpha} = V_\lambda$ where $\lambda = \bigcup_{\alpha<\mu} \lambda_\alpha$.

By TCL, we get

$$V_{\lambda_\alpha} \prec V_\lambda$$

for any $\alpha < \mu$. In particular, $V_\lambda \models ZFC$, so $\lambda$ is worldly.

Since $\mu$ was regular and $\lambda = \bigcup_{\alpha<\mu} \lambda_\alpha$, we have $cf(\lambda) = \mu$.

q.e.d.

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<u>Comment on weakly inaccessible cardinals</u>

The proof that the existence of weakly inaccessible cardinals cannot be proved in ZFC relies on a fundamental theorem by Gödel (his proof of

$$Cons(ZFC) \longrightarrow Cons(ZFC + GCH))$$

This proof is not in the scope of this course (it's one of the Essay topics this year).

# MODELS OF SET THEORY

Suppose $M, N$ are models of set theory s.t. $M \subseteq N$ and they use the same element relation:

$$(M, \in) \quad \text{is a substructure of} \quad (N, \in)$$

[In particular, atomic formulas remain true if you move from $M$ to $N$; as well as if you're going from $N$ to $M$ and your parameters lie in the smaller model.]

We say $M$ is transitive in $N$ if

$$\left.\begin{array}{l} x, y \in N \\ x \in M \\ y \in x \end{array}\right\} \Longrightarrow y \in M$$

[This means that $M$ is a transitive subclass of $N$.]

Example If $\lambda < \kappa$, then $V_\lambda$ is transitive in $V_\kappa$.

**Definition** Let $\Lambda$ be a class of formulas. We say

(i) $\Lambda$ is closed under propositional connectives if
$$\varphi, \psi \in \Lambda \implies \varphi \wedge \psi, \varphi \vee \psi, \neg \varphi \in \Lambda.$$

(ii) $\Lambda$ is closed under bounded quantification if
$$\varphi \in \Lambda \implies \exists x ( x \in y \wedge \varphi ) \in \Lambda$$

$y$: a bound for the variable $x$, $y \neq x$.

**Definition** The class $\Delta_0$ is the smallest class of formulas that contains the atomic formulas and is both closed under propositional connectives and bounded quantification.

**Theorem** Suppose $M$ is transitive in $N$ and $\varphi$ is a $\Delta_0$ formula with $n$ free variables. Assume $\vec{a} \in M^n$.

Then

$$\boxed{M \models \varphi(\vec{a}) \iff N \models \varphi(\vec{a})}$$

Also referred to as ABSOLUTENESS of the formula $\varphi$.

REPHRASING:

$\Delta_0$ formulas are absolute between transitive models of set theory

**Proof.** Since $\Delta_0$ is defined in terms of a recursion (closure), we prove this by induction by proving that the construction steps preserve absoluteness.

- ATOMIC
- PROPOSITIONAL
- BDD QF

1. Atomic: Preserved by all substructures.

2. Propositional connectives: Just the definition of the semantics of $\wedge, \vee, \neg$.

3. Bounded quantification

IH: $\varphi$ is absolute between $M$ and $N$.

Show: $\exists x(x\in y \wedge \varphi)$ is absolute.

"$\Longrightarrow$". Suppose $M \models \underline{\exists x}(x \in a \wedge \varphi)$, $a\in M$

By def. find $b\in M$ s.t.

$M \models b\in a \wedge \varphi$

$\longrightarrow M \models \underline{b\in a}$ and $M\models\varphi$

$\overset{IH}{\Longrightarrow} N \models b\in a$ and $N\models\varphi$

$\Longrightarrow N\models b\in a\wedge\varphi$.

$\Longrightarrow N\models \exists x(x\in a\wedge\varphi)$.

"$\Longleftarrow$" Suppose $N\models\exists x(x\in a\wedge\varphi)$, $\underline{a\in M}$

By def. find $b\in N$ s.t.

$N\models \underline{b\in a}\wedge\varphi$

$N\models b\in a \wedge N\models\varphi$

By transitivity of $M$ in $N$

$b\in M$

$M\models b\in a \wedge M\models\varphi$

$M\models b\in a\wedge\varphi$

$M\models\exists x(x\in a\wedge\varphi)$. q.ed

## Main applications:

Observe that many important notions in set theory are defined by $\Delta_0$ formulas or equivalent to $\Delta_0$ formulas in ZFC.

**Example** "Being an ordinal".

Official definition:

$x$ is an ordinal iff
$x$ is a transitive set s.t.
$(x, \in)$ is a wellorder

[This is not a $\Delta_0$ formula.]

But $ZFC \vdash x$ is an ordinal

$$\Longleftrightarrow$$

$x$ is transitive and
$(x, \in)$ is a total order

$$\forall u,v \, (v \in x \wedge u \in v \longrightarrow v \in x)$$

$$\forall u,v,w \, (u,v,w \in x \longrightarrow (u \in v \wedge v \in w \longrightarrow u \in w) \wedge u \notin u \wedge (u \in v \vee v \in u \vee u = v))$$

<u>Consequence</u> If $M$ is transitive in $N$ and $M \models ZFC$, $N \models ZFC$, then

$x \in M$ $\quad M \models x$ is an ordinal

$$\Longleftrightarrow$$

$N \models x$ is an ordinal.

However, this does not mean

$$\mathrm{Ord} \cap M = \mathrm{Ord} \cap N.$$

[Example: $V_\lambda, V_\kappa$ with $\lambda < \kappa$]

This is an additional property which we discuss in Lecture $\underline{V}$.