LARGE CARDINALS

LECTURE III

31 January 2022

κ WEAKLY INACCESSIBLE if κ regular limit cardinal

κ INACCESSIBLE if κ regular strong limit cardinal

Thm (Hausdorff).

κ inaccessible $\Longrightarrow V_\kappa \models ZFC$

[Even more: V_κ satisfies SOR.]

κ WORLDLY if $V_\kappa \models ZFC$

Goal Show that "inaccessible" is strictly stronger than "worldly".

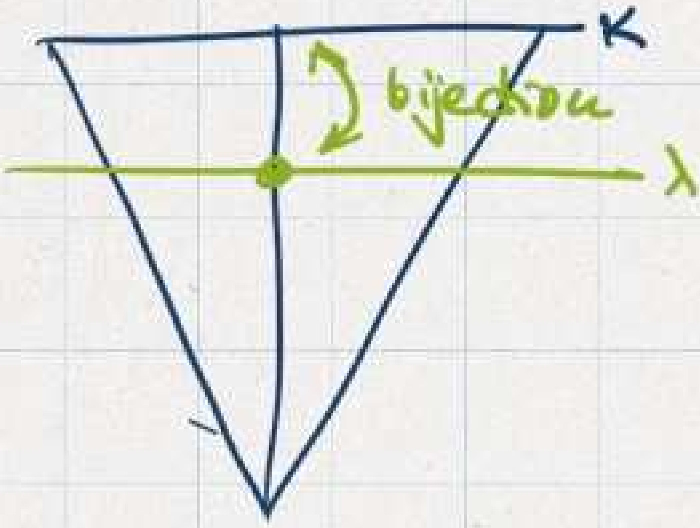
Thm If κ is inaccessible & $\alpha < \kappa$, then there is λ s.t. $\alpha < \lambda < \kappa$ and λ is worldly.

So: if κ is the smallest inaccessible, all worldly cardinals below it are not inaccessible.

First show that $V_\kappa \models ZFC$ implies κ is a cardinal.

<u>Proposition</u> Every worldly κ is a cardinal.

<u>Pf.</u>



[Obviously, κ must be a limit ordinal since every $V_{\alpha+1} \models$ ~~there~~ is a largest ordinal, so $V_{\alpha+1} \not\models ZFC$.]

Use the bijection

$$\pi : \lambda \longrightarrow \kappa$$

to construct a wellorder on λ of o.t. κ in V_κ:

$$R := \{(\alpha, \beta) ; \pi(\alpha) < \pi(\beta)\}$$

By construction, π is an iso between (λ, R) and $(\kappa, \in)$.

But $R \in V_{\lambda+1}$, so $(\lambda, R) \in V_{\lambda+3} \subseteq V_\kappa$.

Since $V_\kappa \models ZFC$, V_κ satisfies the representation thm for wellorders: every wellorder is isomorphic to a unique ordinal.

Thus $\kappa \in V_\kappa$. Contradiction! q.e.d.

Remark A minor improvement of that argument shows:

Every worldly α is a limit cardinal.

[ES #1].

Some basic model theory:

Def Let M, N be $\mathcal{L}$-structures. We write

① $M \equiv N$ — M is elementarily equivalent to N

$:\Longleftrightarrow$ for all $\mathcal{L}$-sentences σ
$M \models \sigma \Longleftrightarrow N \models \sigma$

② $M \preccurlyeq N$ — M is an elementary substructure of N

$:\Longleftrightarrow$ $M \subseteq N$ and for all $\mathcal{L}$-formulas φ with n free variables and all $\vec{a} \in M^n$
$M \models \varphi(\vec{a}) \Longleftrightarrow N \models \varphi(\vec{a})$

Note that $M \equiv N$ is not equivalent to $M \preccurlyeq N$ & $N \preccurlyeq M$.

In fact $M \preccurlyeq N \Longrightarrow M \equiv N$

and in general is much stronger.

Main tool for elementary substructures:

Marker, Model Theory, p. 45

Proposition 2.3.5 (Tarski–Vaught Test) *Suppose that $\mathcal{M}$ is a substructure of $\mathcal{N}$. Then, $\mathcal{M}$ is an elementary substructure if and only if, for any formula $\phi(v, \overline{w})$ and $\overline{a} \in M$, if there is $b \in N$ such that $\mathcal{N} \models \phi(b, \overline{a})$, then there is $c \in M$ such that $\mathcal{N} \models \phi(c, \overline{a})$.*

Sketch the main argument:

[Just induction by complexity of formulas.
Atomic formulas and propositional connectives are preserved for all substructures.
The only missing step is $\varphi \longmapsto \exists x \varphi$ and that step needs precisely the assumption given in TVT.]

Since ZFC is a theory (i.e., consists of sentences), if $V_\kappa \models \mathrm{ZFC}$ and $V_\lambda \equiv V_\kappa$, then $V_\lambda \models \mathrm{ZFC}$ / similarly, $V_\lambda \preccurlyeq V_\kappa$, then $V_\lambda \models \mathrm{ZFC}$.

So, if κ is worldly and $V_\lambda \preccurlyeq V_\kappa$, then λ is worldly.

Thm If κ is inaccessible, then
$$\{\lambda < \kappa \,;\, V_\lambda \preccurlyeq V_\kappa\}$$
is unbounded in κ.

Consequences (1) Since κ is regular, there are κ many worldly cardinals below κ.

(2) The least worldly cannot be the least inaccessible.

Proof of Thm

TVT tells us we need to find witnesses in V_λ for formulas
$$\exists x\, \varphi(x, \vec{a}) \quad \text{where } \vec{a} \in V_\lambda$$
if $V_\kappa \models \exists x\, \varphi(x, \vec{a})$.

Idea: Collect witnesses in an ω-iteration to make the union an elementary substructure by TVT.

Fix $\alpha < \kappa$; try to find λ s.t.
$$\alpha < \lambda < \kappa$$
and $V_\lambda \preccurlyeq V_\kappa$.

Let $\alpha_0 := \alpha + 1$.

Suppose $\underbrace{\alpha_i}_{<\kappa}$ is $\underbrace{\text{already defined}}$.

By our lemma from Lecture II, we know $|V_{\alpha_i}| < \kappa$.

Therefore $|V_{\alpha_i}^{<\omega}| < \kappa$. (*)

the set of finite seq. from V_{α_i}

Furthermore, the set Fml of formulas in the language of set theory $\underbrace{\text{is countable.}}$ (**)

$$(*) + (**) \implies |\mathrm{Fml} \times V_{\alpha_i}^{<\omega}| < \kappa.$$

Suppose $\vec{a} \in V_{\alpha_i}^m$ and $\varphi \in \mathrm{Fml}$ with $m+1$ free variables

$$V_\kappa \models \exists x\, \varphi(x, \vec{a})$$

This can be either true or false.

we do not care, since TVL makes no requirements

<u>Case 1</u> $V_\kappa \not\models \exists x\, \varphi(x, \vec{a})$

Then set

$$w(\varphi, \vec{a}) := 0$$

<u>Case 2</u> $V_\kappa \models \exists x\, \varphi(x, \vec{a})$

Then there is some γ s.t. there is a $c \in V_\gamma$ s.t.

$$V_\kappa \models \varphi(c, \vec{a}).$$

$$w(\varphi, \vec{a}) := \gamma.$$

Since $|\mathrm{Fml} \times (V_{\alpha_i})^{<\omega}| < \kappa$, the set

$$W := \{ w(\varphi, \vec{a}) ;\ \underset{\text{with } n+1 \text{ free var.}}{\varphi \in \mathrm{Fml}} \ \& \ \vec{a} \in (V_{\alpha_i})^n \}$$

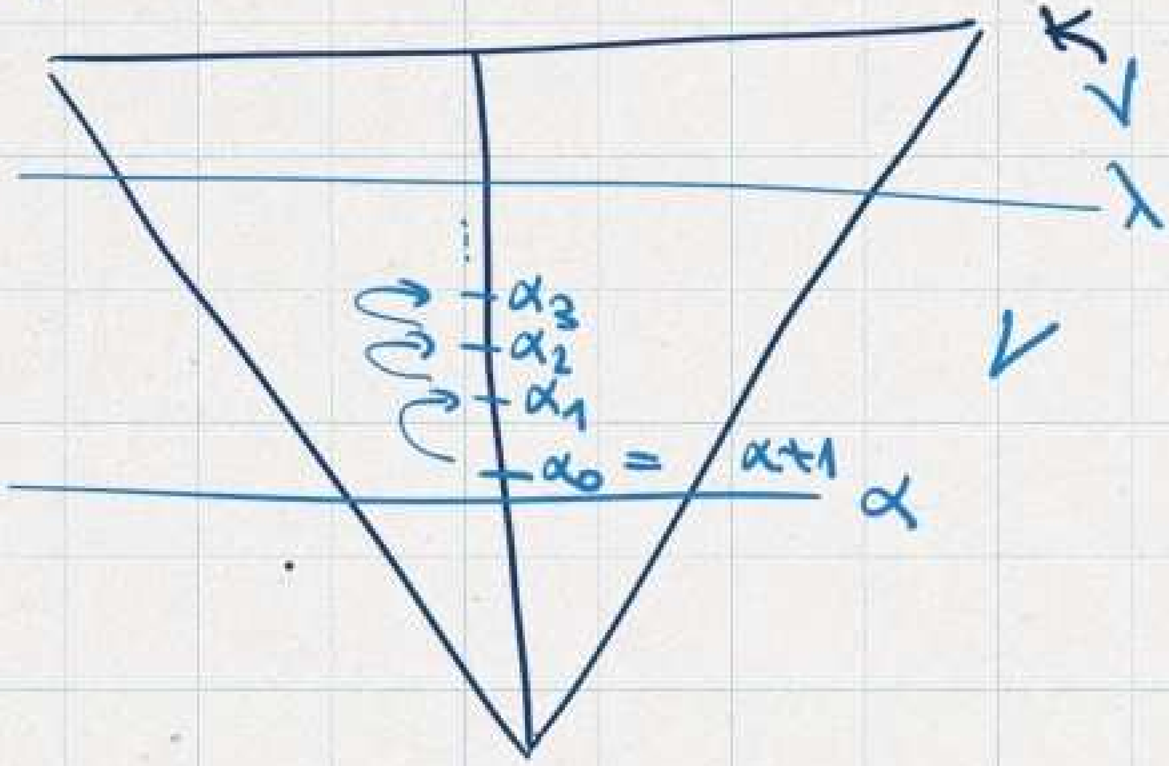
has size $< \kappa$, so by regularity of κ, it is bounded, say, by β.

$$\alpha_{i+1} := \max(\alpha_i + 1, \beta)$$

We define $\lambda := \bigcup_{i \in \mathbb{N}} \alpha_i$.

Clearly $\mathrm{cf}(\lambda) = \aleph_0$.

Therefore $\lambda \neq \kappa$.



<u>Claim</u> $V_\lambda \preccurlyeq V_\kappa$.

For this, we just check that V_λ satisfies the TVT.

Claim $V_\lambda \preccurlyeq V_\kappa$.

Marker, Model Theory, p. 45

Proposition 2.3.5 (Tarski–Vaught Test) *Suppose that $\mathcal{M}$ is a substructure of $\mathcal{N}$. Then, $\mathcal{M}$ is an elementary substructure if and only if, for any formula $\phi(v, \overline{w})$ and $\overline{a} \in M$, if there is $b \in N$ such that $\mathcal{N} \models \phi(b, \overline{a})$, then there is $c \in M$ such that $\mathcal{N} \models \phi(c, \overline{a})$.*

φ has $n+1$ free variables

Suppose * $\boxed{V_\kappa \models \exists x\, \varphi(x, \vec{a})}$ where $\vec{a} \in (V_\lambda)^n$

Since λ is a limit ordinal and $\lambda = \bigcup_{i \in \mathbb{N}} \alpha_i$, we find sufficiently large N s.t.

$$\vec{a} \in (V_{\alpha_N})^n.$$

So * means that we're in Case 2 of the construction in the step $\alpha_N \longmapsto \alpha_{N+1}$.

Therefore, there is a witness $c \in V_{\alpha_{N+1}}$ s.t.

$$V_\kappa \models \varphi(c, \vec{a}).$$

This shows the TVT.

q.e.d.

<u>Remarks</u> As seen in the proof,

$$\mathrm{cf}(\lambda) = \aleph_0.$$

That means:

① The least such worldly cardinal has cofinality $\aleph_0$.

② The proof shows that

$$W^{\aleph_0} := \{\lambda < \kappa ; V_\lambda \preccurlyeq V_\kappa \ \& \ \mathrm{cf}(\lambda) = \aleph_0\}$$

is unbounded in κ.

If a worldly cardinal is regular, then it is weakly inaccessible (by ES#1).

This means there is no hope to show that any of these $\lambda < \kappa$ are regular:

[If GCH holds, for all $\lambda < \kappa$, $2^\lambda = \lambda^+$.
Thus "strong limit" $\Longleftrightarrow$ "limit" and therefore "weakly inacc." $\Longleftrightarrow$ "inacc.".
So in ZFC+GCH there are inacc. cardinals with no regular worldlies below.]

<u>Q</u>: Can we get worldlies with higher cofinality, e.g., $\mathrm{cf}(\lambda) = \aleph_1, \aleph_2$ etc.