LARGE CARDINALS
SECOND LECTURE

26 January 2022

INFORMAL DEFINITION

A large cardinal property is a property such that
(a) it implies being very big
(b) the existence of such cardinals cannot be proved

EXAMPLES

I. Aleph fixed points $\kappa = \aleph_\kappa$ satisfy (a), not (b)

II. Weakly inaccessibles (= regular limit cardinals)

(Strongly) inaccessibles (= regular strong limit cardinals) $\forall \lambda < \kappa \; (2^\lambda < \kappa)$

$\mathrm{IC}(\kappa) :\Longleftrightarrow \kappa$ is regular & a strong limit

$\mathrm{IC} \; :\Longleftrightarrow \exists \kappa \; \mathrm{IC}(\kappa)$

GOAL: $\mathrm{ZFC} \nvdash \mathrm{IC}$.

Reminder: von Neumann hierarchy / cumulative hierarchy

$$V_0 := \emptyset$$
$$V_{\alpha+1} := \mathcal{P}(V_\alpha)$$
$$V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha \qquad \lambda \text{ limit ordinal}$$

Properties:
- all V_α are transitive sets
- $\alpha \leq \beta \Longrightarrow V_\alpha \subseteq V_\beta$ ["cumulative"]
- $V_\alpha \cap \mathrm{Ord} = \alpha$

A theorem that was proved:
The axiom of Foundation / Regularity is equivalent to

$$\forall x \exists \alpha \quad x \in V_\alpha$$

MIRIMANOFF rank:

$$\rho(x) := \min\{\alpha;\ x \in V_{\alpha+1}\}$$

In ZFC, each set has a Mirimanoff rank and we can prove statements about all sets by induction over the rank.

REMINDER

Lent 2021 **LOGIC AND SET THEORY – EXAMPLES 4** PAR

10. A set x is called *hereditarily finite* if each member of $\mathrm{TC}(\{x\})$ is finite. Prove that the class HF of hereditarily finite sets coincides with V_ω. Which of the axioms of ZF are satisfied in the structure HF?

11. Which of the axioms of ZF are satisfied in the structure $V_{\omega+\omega}$?

V_ω

$V_{\omega+\omega}$

Observation: Most of the ZFC axioms are of the form $\forall x \exists y$, so proving such an axiom in $(V_\alpha, \in)$ is the same as proving a bound on ρ for y in terms of x.

ES#4.10: If λ is a limit ordinal, then $(V_\lambda, \in)$ satisfies all axioms of ZFC with the exception of Infinity & Replacement

ES#4.11: If $\lambda > \omega$, then Infinity is true in $(V_\lambda, \in)$.

SUMMARY: If $\lambda > \omega$ is a limit ordinal, then $(V_\lambda, \in) \models ZC$ Zermelo set theory with choice [all of ZFC without Replacement]

Theorem (Hausdorff).

If κ is inaccessible, then

$$V_\kappa \models \mathrm{ZFC}.$$

[By the previous reminder, this means that

$V_\kappa \models$ Replacement.]

Corollary. If ZFC is consistent, then

$$\mathrm{ZFC} \nvdash \mathrm{IC}.$$

Proof. Suppose $\mathrm{ZFC} \vdash \mathrm{IC}$. (*)

Hausdorff says: $\mathrm{IC} \Longrightarrow \exists M (M \models \mathrm{ZFC})$

$$(**) \qquad \underset{\substack{\text{Gödel's}\\ \text{Completeness}\\ \text{Thm}}}{\Longleftrightarrow} \mathrm{Cons}(\mathrm{ZFC})$$

Modus ponens to (*) & (**) implies

$$\mathrm{ZFC} \vdash \mathrm{Cons}(\mathrm{ZFC}).$$

By Gödel's second incompleteness theorem, this implies that ZFC is inconsistent.

q.e.d.

Lemma If $IC(\kappa)$ and $\alpha < \kappa$, then $|V_\alpha| < \kappa$.

Proof. By induction:

(0) $|V_0| = 0 < \kappa$. ✓

($\alpha \to \alpha+1$) Suppose $|V_\alpha| < \kappa$.

Then $|V_{\alpha+1}| = |\mathcal{P}(V_\alpha)|$

$$= 2^{|V_\alpha|} < \kappa.$$

κ strong limit $\iff$ $\forall \lambda (\lambda < \kappa \Rightarrow 2^\lambda < \kappa)$

(λ limit) Suppose for all $\alpha < \lambda$ $|V_\alpha| < \kappa$.

$$|V_\lambda| = \left|\bigcup_{\alpha<\lambda} V_\alpha\right| \leq \bigcup_{\alpha<\lambda} |V_\alpha|$$

$\{|V_\alpha| ; \alpha < \lambda\}$ is a set of ordinals in κ with size $\leq |\lambda| < \kappa$

So by regularity: $\bigcup_{\alpha<\lambda} |V_\alpha|$ is bounded below κ.

So $|V_\lambda| < \kappa$. q.e.d.

<u>Corollary</u> If $IC(\kappa)$ and $x \in V_\kappa$, then $|x| < \kappa$.

<u>Pf.</u> If $x \in V_\kappa$, there is $\alpha < \kappa$ s.t. $x \in V_\alpha$. By transitivity, $x \subseteq V_\alpha$

$$|x| \leq |V_\alpha| \underset{\text{Lemma}}{<} \kappa.$$

q.e.d.

<u>PROOF OF HAUSDORFF'S THEOREM</u>

By earlier discussion, we only need to prove $V_\kappa \models$ Replacement.

We strengthen Replacement to the following principle:

SOR

SECOND ORDER REPLACEMENT

$\boxed{\begin{array}{l}\text{for all } F: V_\kappa \longrightarrow V_\kappa \text{ and} \\ \text{all } x \in V_\kappa, \text{ the image} \\ F[x] := \{F(y); y \in x\} \in V_\kappa.\end{array}}$

Replacement is the restriction of SOR to F that are definable over V_κ by a first-order formula.

For now, it only matters that if V_κ satisfies SOR then $V_\kappa \models$ Replacement. [More discussion later.]

Let's prove SOR:

Fix $F : V_\kappa \longrightarrow V_\kappa$,

fix $x \in V_\kappa$.

Show: $\boxed{F[x] \in V_\kappa.}$

$$F[x] = \{\underset{\in V_\kappa}{F(y)};\ y \in x\}$$

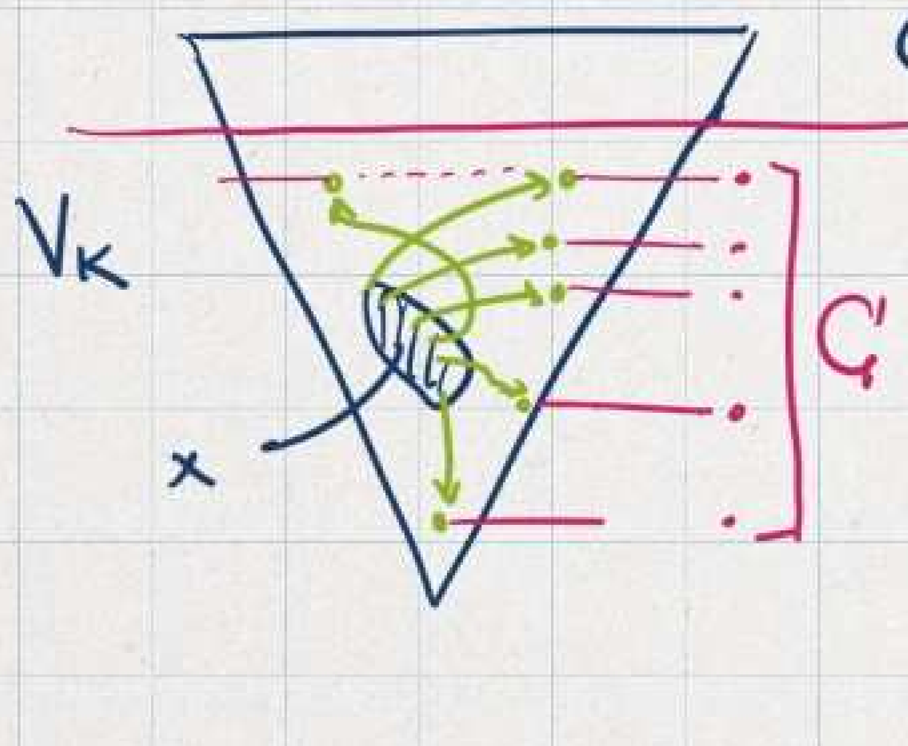
So $\rho(F(y)) < \kappa$.

$$C := \{\rho(F(y));\ y \in x\} \subseteq \kappa$$

$$|C| \leq |x| < \kappa$$

[by Corollary]

So by regularity of κ,



C is bounded, say by α.

So: $F[x] \subseteq V_\alpha$,

so $F[x] \in V_{\alpha+1} \subseteq V_\kappa$.

q.e.d.

DISCUSSION

1. We can prove $ZFC \nvdash IC$ from Hausdorff's theorem without invoking Gödel's Second Incompleteness Theorem.

<u>Sketch</u> First need to show that if κ is inaccessible and $\lambda < \kappa$, then

$$(*) \qquad IC(\lambda) \iff V_\kappa \models IC(\lambda)$$

[Example Sheet #1]

(**) Assume $ZFC \vdash IC$.

Let κ_0 be the least inaccessible.

By Hausdorff, $V_{\kappa_0} \models ZFC$.

By (**), $V_{\kappa_0} \models IC$.

Therefore, there is $\lambda < \kappa_0$ s.t.

$$V_{\kappa_0} \models IC(\lambda).$$

$$\overset{(*)}{\iff} IC(\lambda).$$

in contradiction to the minimality of κ_0.

2. The proof of Hausdorff's theorem seems to suggest that everything fits so nicely that it should be an equivalence!

<u>Def.</u> κ is called <u>WORLDLY</u> if
$$V_\kappa \models ZFC.$$

Hausdorff says: Every inaccessible is worldly.

<u>Q.</u> Are these notions equivalent?
Is Hausdorff's theorem an equivalence?

<u>A</u>: NO/YES.

<u>Yes</u>: The equivalence is not with worldliness, but rather with the stronger property SOR. [Example Sheet #1 and further lectures.]

<u>No</u>: If κ is inaccessible, there are many worldly cardinals that are not inaccessible.

<u>Theorem</u> If κ is inaccessible and $\alpha<\kappa$, then there is λ s.t. $\alpha<\lambda<\kappa$ and λ is worldly.