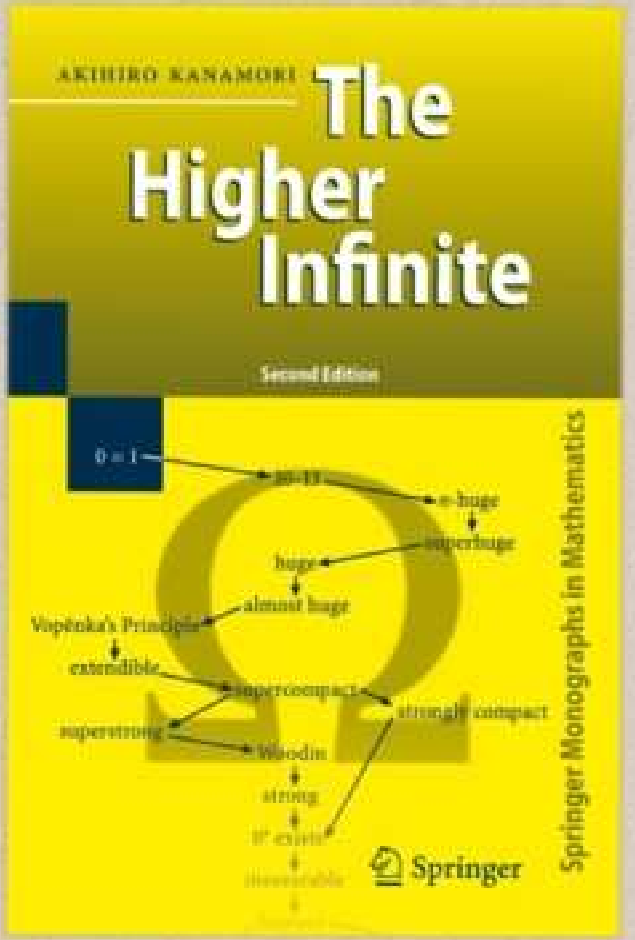
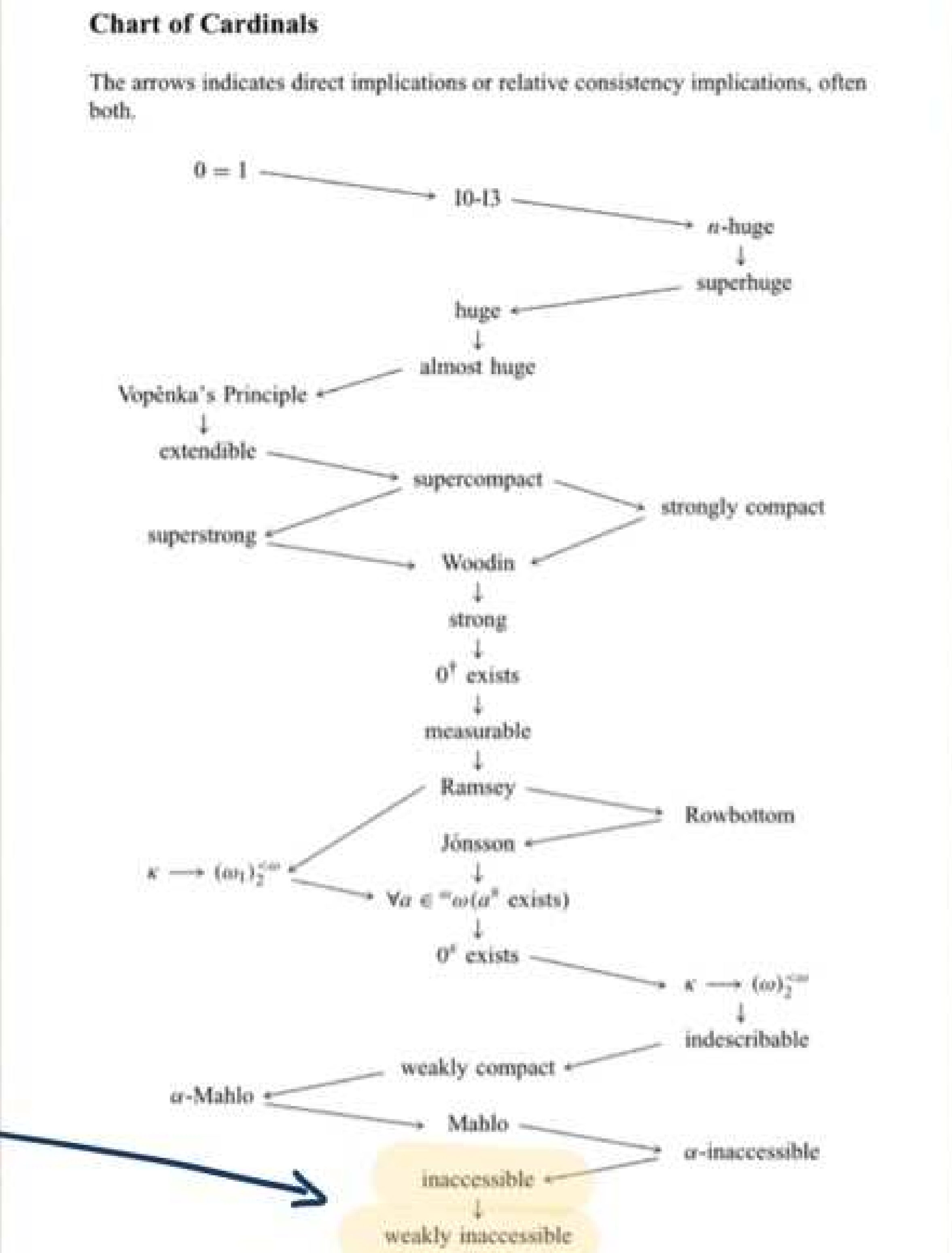
LARGE CARDINALS

LENT TERM 2022

Lecture I

24 January 2022





Kanamori, The Higher Infinite, p. 472

TODAY

weakly inaccessibles/

(strongly) inaccessibles

NON-DEFINITIONS

1. A **LARGE CARDINAL PROPERTY** IS A PROPERTY OF A CARDINAL κ THAT IMPLIES THAT
   (a) κ IS VERY BIG
   (b) SO BIG THAT ZFC CANNOT PROVE THE EXISTENCE OF SUCH CARDINALS
2. A **LARGE CARDINAL** IS A CARDINAL THAT HAS A LARGE CARDINAL PROPERTY.
3. A **LARGE CARDINAL AXIOM** IS AN AXIOM OF THE FORM

$$\exists \alpha\, \Phi(\alpha)$$

WHERE Φ IS A LARGE CARDINAL PROPERTY.

Today: Understanding Non-Definition 1.

(a) $\longrightarrow$ very big

(b) $\longrightarrow$ too big for ZFC.

EXAMPLE I

A property that implies "κ is very big"?

Reminder: NORMAL ORDINAL OPERATIONS

$$F : \mathrm{Ord} \longrightarrow \mathrm{Ord}$$

[not a set, a class behaving like a function]

is called NORMAL ORDINAL OPERATION if it is

monotone $\quad \alpha < \beta \Longrightarrow F(\alpha) < F(\beta)$

and

continuous $\quad$ if λ limit ordinal, then $F(\lambda) = \bigcup_{\alpha<\lambda} F(\alpha)$

<u>Theorem</u> (essentially from L&ST)

Every normal ordinal operation has arbitrarily large fixed points: $\xi = F(\xi)$.

Reminder.

Lent 2021 **LOGIC AND SET THEORY—EXAMPLES 2** PAR

10. What is the smallest fixed point of $\alpha \mapsto \omega^\alpha$? The next smallest? And the next smallest? Show that the fixed points are unbounded, and explain why this means that we may index the fixed points by the ordinals. Is there a countable ordinal α such that α is the α-th fixed point?

[This will be discussed in more detail on ES#1 for LC.]

The following is a normal ordinal operation:

$$\aleph_0 := \omega$$

$$\aleph_{\alpha+1} := \text{the card. succ. of } \aleph_\alpha$$

$$\aleph_\lambda := \bigcup_{\alpha<\lambda} \aleph_\alpha$$

By the theorem, there are arbitrarily large fixed pts:

$$\kappa = \aleph_\kappa$$

Called Aleph fixed pts.

Aleph fixed pts must be very large:

$$\kappa = \aleph_\kappa$$

Since κ is a cardinal, κ is a limit ordinal, so $\aleph_\kappa = \kappa$ is a limit cardinal.

$\aleph_\omega \neq \omega$ (because $\omega \neq 0$)

$\aleph_{\omega_1} \neq \aleph_1$ (because $\omega_1 \neq 1$)

$\aleph_{\omega_2} \neq \aleph_2$ (because $\omega_2 \neq 2$)

GAP between these indices is growing.

$\aleph_{\aleph_\omega} \neq \aleph_\omega$ (because $\aleph_\omega \neq \omega$)

$\vdots$

Therefore, the first aleph fixed pt (which is $\aleph_{\aleph_{\aleph_{\ddots}}} = \bigcup_{n \in \mathbb{N}} \underbrace{\aleph_{\aleph_{\dots}}}_{n \text{ times}} 0$)

is much bigger than all of these.

So: it is "very big".

But: it's not

"too big for ZFC".

If $AFP(\kappa) :\Longleftrightarrow \kappa$ is an aleph fixed pt.

and $AFP :\Longleftrightarrow \exists \kappa\, AFP(\kappa)$,

then $ZFC \vdash AFP$.

[That's precisely the theorem from page 3.]

So AFP is <u>not</u> a large cardinal property in the sense of (b).

EXAMPLE II

We need the notion of COFINALITY.
Let λ be a limit ordinal. Then $C \subseteq \lambda$ is called COFINAL or UNBOUNDED if

$$\forall \alpha < \lambda \, \exists \gamma \in C \, (\alpha < \gamma).$$

Equivalently, $\lambda = \bigcup C$.

Clearly, λ is unbounded in λ. (*)

<u>Def.</u> We define the <u>COFINALITY of λ</u>

$$\mathrm{cf}(\lambda) := \min\{|C| ;\ C \text{ is cofinal in } \lambda\}$$

(*) means $\mathrm{cf}(\lambda) \leq |\lambda|$.
Obviously, $\mathrm{cf}(\lambda)$ is a cardinal. So if λ is not a cardinal, $\mathrm{cf}(\lambda) \leq |\lambda| < \lambda$.

<u>Reminder</u>. ES#2 Example 9 L&ST asked you to show that ω_1 is not a countable union of countable ordinals.
So in our terminology: $\mathrm{cf}(\omega_1) = \omega_1$.

For cardinal κ, $cf(\kappa) \leq \kappa$.

Def. A cardinal is called REGULAR if $cf(\kappa) = \kappa$; it is called SINGULAR if $cf(\kappa) < \kappa$.

Example 1. $\aleph_\omega$ is singular.

$$\aleph_\omega = \bigcup \underbrace{\{\aleph_n ; n \in \mathbb{N}\}}_{\text{so this set is cofinal}}$$

$$\Longrightarrow cf(\aleph_\omega) \leq \aleph_0$$

[and therefore $cf(\aleph_\omega) = \aleph_0$]

In general, if $\kappa = \aleph_\lambda$ is a limit cardinal (i.e., λ is a limit ordinal), then

$$\kappa = \aleph_\lambda = \bigcup \underbrace{\{\aleph_\alpha ; \alpha < \lambda\}}_{\text{is cofinal}}$$

$$cf(\aleph_\lambda) \leq cf(\lambda).$$

Example 2. We had already discussed that $\aleph_1$ is regular (ES#2, Example 9).

Generalise this to:

<u>Theorem (ZFC).</u> Every successor cardinal is regular.

<u>Proof.</u> Let $\kappa = \aleph_{\alpha+1}$.

By definition, if $\xi < \kappa$, there is a surjection from $\aleph_\alpha$ onto ξ. Use AC to pick such a surjection

$$\pi_\xi : \aleph_\alpha \twoheadrightarrow \xi$$

for each ξ.

Towards a contradiction, assume

$$\kappa = \bigcup C \qquad (*)$$

where C is of size $\leq \aleph_\alpha$.

So fix surjection $\psi : \aleph_\alpha \twoheadrightarrow C$.

Define

$$\begin{aligned} \pi : \boxed{\aleph_\alpha \times \aleph_\alpha} &\longrightarrow \aleph_{\alpha+1} \\ (\gamma, \delta) &\longmapsto \pi_{\psi(\gamma)}(\delta). \end{aligned}$$

By (*), π is a surjection.

However, by Hessenberg's Thm

$\aleph_\alpha \times \aleph_\alpha$

is in bijection with $\aleph_\alpha$, so we found a surjection from $\aleph_\alpha$ onto $\aleph_{\alpha+1}$. Contradiction!

q.e.d.

	REGULAR	SINGULAR
SUCCESSOR	Every.	×
LIMIT	?	All concrete examples we could come up with.

<u>Def.</u> A cardinal is called <u>weakly inaccessible</u> if it is a regular limit cardinal.

Show property (a): very big!

<u>Observation</u> Each weakly inaccessible cardinal is an Aleph fixed point.

<u>Proof.</u> κ is weakly inacc.

$[\Rightarrow$ limit cardinal$]$

$\kappa = \aleph_\lambda$ where λ is a limit ordinal.

We saw earlier that $\underline{cf(\kappa)} = cf(\aleph_\lambda) \leq \underline{cf(\lambda)}$.
$\leq \lambda.$

So if $\lambda < \kappa$, then $\underline{cf(\kappa) \leq \lambda < \kappa}$,
so κ is singular.

So $\lambda = \kappa$, and κ is an aleph fixed pt.

q.e.d.

At the moment, we cannot show that

ZFC $\vdash$ there are weakly inaccessibles

[that would be (b)]

We will be able to show this for a slight strengthening:

<u>Def.</u> • A cardinal κ is called (STRONGLY) INACCESSIBLE if it is a regular strong limit cardinal.

• A cardinal κ is called a STRONG LIMIT if for each $\lambda < \kappa$,

$$2^{\lambda} < \kappa.$$

[Compare: κ is a limit cardinal
$$\Longleftrightarrow$$
$\forall \alpha$ s.t. $\aleph_\alpha < \kappa$, $\aleph_{\alpha+1} < \kappa$]

In lecture II, we're going to show Hausdorff's Thm: [If ZFC is consistent, then]

$$\mathrm{ZFC} \nvdash \text{there is an inaccessible.}$$