Model Solutions for Example Sheet #3

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(31) Let κ be inaccessible and L be any $\mathcal{L}_{\kappa\kappa}$ language and M an L-structure. Write L^α for the set of L-formulas whose free variables are contained in $\{v_\xi\,;\,\xi<\alpha\}$. If $X\subseteq M$, we say that X is an L-*elementary substructure* (in symbols: $X\preccurlyeq_L M$) if for all $\alpha<\kappa$, $\varphi\in L^\alpha$ and all $\vec{x}\in X^\alpha$, we have that

$$X\frac{\vec{x}}{\vec{v}}\models\varphi \iff M\frac{\vec{x}}{\vec{v}}\models\varphi.$$

Prove the following statement (*Tarski-Vaught Test for $\mathcal{L}_{\kappa\kappa}$ languages*): a subset X is an L-elementary substructure if and only if it is an L-substructure and for all $\alpha,\beta<\kappa$, $\varphi(\vec{v},\vec{w})\in L^{\alpha+\beta}$ (with $\vec{v}:=\{v_\xi\,;\,\xi<\alpha\}$ and $\vec{w}:=\{v_{\alpha+\eta}\,;\,\eta<\beta\}$) and all $\vec{x}\in X^\alpha$, if $M\frac{\vec{x}}{\vec{v}}\models\exists^\beta\vec{w}\varphi$, then there is some $\vec{y}\in X^\beta$ such that $M\frac{\vec{x}}{\vec{v}}\frac{\vec{y}}{\vec{w}}\models\varphi$. (Why do we require the inaccessibility of κ?)

Solution. Let $X\subseteq M$ be L-structures and suppose that X is an L-elementary substructure of M. If $M\frac{\vec{x}}{\vec{v}}\models\exists^\beta\vec{w}\varphi$ for $\vec{x}\in X^\alpha$ then by elementarity $X\frac{\vec{x}}{\vec{v}}\models\exists^\beta\vec{w}\varphi$. Hence by definition there is $\vec{y}\in X^\beta$ such that $X\frac{\vec{x}}{\vec{v}}\frac{\vec{y}}{\vec{w}}\models\varphi$, and so $M\frac{\vec{x}}{\vec{v}}\frac{\vec{y}}{\vec{w}}\models\varphi$ again by elementarity.

Conversely, we argue by induction on the structure of $\varphi\in L^{\alpha+\beta}$ that the embedding is elementary. For φ quantifier-free the argument is essentially the same as in the standard Tarski-Vaught test. So, suppose that $\beta<\kappa$ and $\varphi=\exists^\beta\vec{w}\vartheta$, for $\vartheta\in L^{\alpha+\beta}$. If $X\frac{\vec{x}}{\vec{v}}\models\exists^\beta\vec{w}\vartheta$ then by definition there is $\vec{y}\in X^\beta$ such that $X\frac{\vec{x}}{\vec{v}}\frac{\vec{y}}{\vec{w}}\models\vartheta$, and so by the induction hypothesis $M\frac{\vec{x}}{\vec{v}}\frac{\vec{y}}{\vec{w}}\models\vartheta$, i.e. $M\frac{\vec{x}}{\vec{v}}\models\exists^\beta\vec{w}\vartheta$. If on the other hand $M\frac{\vec{x}}{\vec{v}}\models\exists^\beta\vec{w}\vartheta$ then by the assumption there is some $\vec{y}\in X^\beta$ such that $M\frac{\vec{x}}{\vec{v}}\frac{\vec{y}}{\vec{w}}\models\vartheta$. By the induction hypothesis we have that $X\frac{\vec{x}}{\vec{v}}\frac{\vec{y}}{\vec{w}}\models\vartheta$, and so $X\frac{\vec{x}}{\vec{v}}\models\exists^\beta\vec{w}\vartheta$.

We implicitly use the regularity of κ in the assumption that the variables of the formulas we induct on are bounded in κ. In particular, if $\lambda<\kappa$ and $(\phi_\beta)_{\beta<\lambda}$ are such that each ϕ_β is in $L^{f(\beta)}$ for some $f:\lambda\to\kappa$, then $\bigvee_{\beta<\lambda}\phi_\beta$ is in some L^α since f ought to be bounded by the regularity of κ.

$\dashv$

(32) Let κ be inaccessible, L be any $\mathcal{L}_{\kappa\kappa}$ language, M an L-structure, and $X\subseteq M$ of size $\leq\kappa$. If $\varphi\in L^{\alpha+\beta}$ (with $\vec{v}:=\{v_\xi\,;\,\xi<\alpha\}$ and $\vec{w}:=\{v_{\alpha+\eta}\,;\,\eta<\beta\}$) and and $\vec{x}\in M^\alpha$ such that $M\frac{\vec{x}}{\vec{v}}\models\exists^\beta\vec{w}\varphi$, then there is some $\vec{y}\in M^\beta$ such that $M\frac{\vec{x}}{\vec{v}}\frac{\vec{y}}{\vec{w}}\models\varphi$. Use the Axiom of Choice to assign such a witness $\mathrm{w}(\varphi,\vec{x})$. Let $H(X,\alpha):=X\cup\bigcup\{\mathrm{Im}(\mathrm{w}(\varphi,\vec{x}))\,;\,\varphi\in L^{\alpha+\beta},\,\vec{x}\in X^\alpha\}$. Define by recursion $H_0(X):=X$, $H_{\alpha+1}(X):=H(H_\alpha(X),\alpha)$, and $H_\lambda(X):=\bigcup_{\alpha<\lambda}H_\alpha(X)$ (for limit ordinals λ) and show that $H_\kappa(X)\preccurlyeq_L M$ is an elementary substructure of cardinality $\leq\kappa$.

Solution. Define $w:L^{\alpha+\beta}\times M^\alpha\to M^\beta$ in the following way: if $M\frac{\vec{x}}{\vec{v}}\models\exists^\beta\vec{w}\varphi$ then pick (using Choice) a witness $\vec{y}\in M^\beta$ such that $M\frac{\vec{x}}{\vec{v}}\frac{\vec{y}}{\vec{w}}\models\varphi$; if $M\frac{\vec{x}}{\vec{v}}\not\models\exists^\beta\vec{w}\varphi$ then pick a fixed $\vec{m_0}\in M^\beta$. Let $H_\kappa(X)$ be as above. We use the Tarski-Vaught test for $\mathcal{L}_{\kappa\kappa}$ languages to argue that $H_\kappa(X)\preccurlyeq_L M$. Indeed, suppose that $M\frac{\vec{x}}{\vec{v}}\models\exists^\beta\vec{w}\varphi$ for some $\varphi\in L^{\alpha+\beta}$ and $\vec{x}\in H_\kappa(X)^\alpha$. Since $\alpha<\kappa$, there is by regularity some $\gamma<\kappa$ such that $\vec{x}\in H_\gamma(X)^\alpha$. It follows by construction that $\vec{y}=w(\phi,\vec{x})$ is such that $M\frac{\vec{x}}{\vec{v}}\frac{\vec{y}}{\vec{w}}\models\varphi$, and furthermore $\vec{y}\in H_{\gamma+1}(X)^\alpha\subseteq H_\kappa(X)^\alpha$. It follows by Tarski-Vaught that $H_\kappa(X)\preccurlyeq_L M$. Furthermore, assuming that $|X|\leq\kappa$, we may show by induction on $\alpha<\kappa$ that $|H_\alpha(X)|\leq\kappa$, and hence $|H_\kappa(X)|\leq\kappa$. $\dashv$

(33) Show that the consistency strength hierarchy has the following properties:

(a) $0 = 1$ is maximal w.r.t. $\leq_{\text{Cons}}$;

(b) if A is not maximal, then there is B such that $\mathsf{A} <_{\text{Cons}} \mathsf{B}$ and B is not maximal;

(c) for all A and B, if $\mathsf{A} \leq_{\text{Cons}} \mathsf{B}$, then $\mathsf{A} \vee \mathsf{B} \equiv_{\text{Cons}} \mathsf{A}$.

Solution.

(a) Since $0 = 1$ proves anything, we see that $\text{Cons} \cap C_{\mathsf{ZFC}+0=1} = \text{Cons}$, and so for any sentence A, $\text{Cons} \cap C_{\mathsf{ZFC+A}} \subseteq \text{Cons}$, i.e. $\mathsf{A} \leq_{\text{Cons}} 0 = 1$. In fact, it is easy to see that A is maximal if and only if $\mathsf{ZFC} + \mathsf{A}$ is inconsistent.

(b) Suppose that A is not maximal. Consider $\mathsf{B} = \mathsf{A} + \text{Cons}(\mathsf{ZFC} + \mathsf{A})$. Since A is not maximal it follows that $\mathsf{ZFC}+\mathsf{A}$ is consistent, and so B is consistent and therefore not maximal. Furthermore, $\mathsf{B} \implies \mathsf{A}$ and so $\mathsf{A} \leq_{\text{Cons}} B$, while $\mathsf{ZFC} + \mathsf{B} \vdash \text{Cons}(\mathsf{ZFC} + \mathsf{A})$ and $\mathsf{ZFC} + \mathsf{A} \nvdash \text{Cons}(\mathsf{ZFC} + \mathsf{A})$. It follows that $\mathsf{A} <_{\text{Cons}} B$.

(c) Suppose that $\mathsf{A} \leq_{\text{Cons}} \mathsf{B}$. Clearly $A \implies \mathsf{A} \vee \mathsf{B}$, and therefore $\mathsf{A} \vee \mathsf{B} \leq_{\text{Cons}} \mathsf{A}$. Furthermore, if $\mathsf{ZFC}+\mathsf{A} \vdash \text{Cons}(\mathsf{ZFC}+\mathsf{C})$, then $\mathsf{ZFC}+\mathsf{B} \vdash \text{Cons}(\mathsf{ZFC}+\mathsf{C})$ by the assumption. Hence $\mathsf{ZFC}+(\mathsf{A}\vee\mathsf{B}) \vdash \text{Cons}(\mathsf{ZFC} + \mathsf{C})$, i.e. $\mathsf{A} \leq_{\text{Cons}} \mathsf{A} \vee \mathsf{B}$.

$\dashv$

(34) Let Φ be a cardinal property (i.e., $\Phi(\kappa)$ implies that κ is a cardinal). Let us say that Φ is *nontrivial* if $\Phi(\kappa)$ implies that κ is inaccessible. Show that there is a nontrivial Φ such that $\Phi\mathsf{C} \equiv_{\text{Cons}} \mathsf{IC}$ and $\mathsf{WC} <_1 \Phi\mathsf{C}$. Use this to argue that the following statement is in general false: if $\mathsf{A} \leq_{\text{Cons}} \mathsf{B}$, then $\mathsf{A} \wedge \mathsf{B} \equiv_{\text{Cons}} \mathsf{B}$.

Solution. Let $\Phi(\kappa) =$ "κ is inaccessible $\wedge(\mathsf{WC} \to \kappa$ is larger than the least w.c.)". Clearly $\mathsf{ZFC}+\Phi\mathsf{C} \vdash \mathsf{IC}$, so $\mathsf{IC} \leq_{\text{Cons}} \Phi\mathsf{C}$. Conversely, suppose that $\mathsf{ZFC}+\Phi\mathsf{C} \vdash \text{Cons}(\mathsf{ZFC}+\varphi)$. Clearly if $\mathsf{ZFC}+\mathsf{IC}$ is inconsistent then $\mathsf{ZFC} + \mathsf{IC} \vdash \text{Cons}(\mathsf{ZFC} + \varphi)$ as it proves everything. If $\mathsf{ZFC} + \mathsf{IC}$ is consistent, then fix some $M \models \mathsf{ZFC} + \mathsf{IC}$. If $M \models \neg\mathsf{WC}$ then $M \models \Phi\mathsf{C}$ so $M \models \text{Cons}(\mathsf{ZFC} + \varphi)$. If on the other hand $M \models \mathsf{WC}$ then fix some $\kappa \in M$ such that $M \models$ "κ is the least weakly compact". By transfinite recursion in M we may define $\mathbf{V}_\kappa^M$, which is transitive in M. Since there are inaccessibles below the least weakly compact, $M \models$ "$\mathbf{V}_\kappa \models \mathsf{IC}+\neg WC$", and so in fact $M \models$ "$\mathbf{V}_\kappa \models \mathsf{ZFC}+\Phi\mathsf{C}$". Hence $M \models$ "$\mathbf{V}_\kappa \models \text{Cons}(\mathsf{ZFC}+\varphi)$", and since this is an arithmetical (so Δ_0) statement $M \models \text{Cons}(\mathsf{ZFC} + \varphi)$. It follows by completeness that $\mathsf{ZFC} + \mathsf{IC} \vdash \text{Cons}(\mathsf{ZFC} + \varphi)$, and so $\mathsf{ZFC} + \Phi\mathsf{C} \equiv_{\text{Cons}} \mathsf{ZFC} + \mathsf{IC}$.

Also, $\mathsf{WC} <_1 \Phi\mathsf{C}$. Indeed, assuming $\mathsf{WC} + \Phi\mathsf{C}$ then the least weakly compact cardinal is by definition strictly smaller than the least cardinal satisfying $\Phi(\kappa)$.

Observe that $\Phi\mathsf{C} \equiv_{\text{Cons}} \mathsf{IC} \leq_{\text{Cons}} \mathsf{WC}$. However, $\Phi\mathsf{C} \wedge \mathsf{WC} >_{\text{Cons}} \mathsf{WC}$. Indeed, in $\mathsf{ZFC} + \Phi\mathsf{C} + \mathsf{WC}$ we may find $\lambda < \kappa$ with λ the least weakly compact and κ inaccessible. But then $\mathbf{V}_\kappa \models \mathsf{ZFC} + \mathsf{WC}$, so $\mathsf{ZFC} + \Phi\mathsf{C} + \mathsf{WC} \vdash \text{Cons}(\mathsf{ZFC} + \mathsf{WC})$. On the other hand $\mathsf{ZFC} + \mathsf{WC} \nvdash \text{Cons}(\mathsf{ZFC} + \mathsf{WC})$ by Gödels 2nd Incompleteness Theorem. $\dashv$

(35) Let A be the statement "if there is a weakly compact cardinal κ, then there is an inaccessible $\lambda > \kappa$". Show that the consistency strength of $\mathsf{ZFC}+\mathsf{A}$ is equal to that of ZFC, but that under some consistency assumptions, $\mathsf{ZFC} <_0 \mathsf{ZFC} + \mathsf{A}$. What are the required consistency assumptions for the latter claim?

Solution. Clearly $C_{\mathsf{ZFC}} \subseteq C_{\mathsf{ZFC+A}}$. For the other direction we repeat the argument in (34). Suppose that $\mathsf{ZFC} + \mathsf{A} \vdash \text{Cons}(\mathsf{ZFC} + \varphi)$. Clearly if ZFC is inconsistent then $\mathsf{ZFC} \vdash \text{Cons}(\mathsf{ZFC} + \varphi)$ as it proves everything. If ZFC is consistent, then fix some $M \models \mathsf{ZFC}$. If $M \models \neg\mathsf{WC}$ then $M \models \mathsf{A}$ so $M \models \text{Cons}(\mathsf{ZFC} + \varphi)$. If on the other hand $M \models \mathsf{WC}$ then fix some $\kappa \in M$ such that $M \models$ "κ is the least weakly compact". By transfinite recursion in M we may define $\mathbf{V}_\kappa^M$, which is transitive in M. But then $M \models$ "$\mathbf{V}_\kappa \models \neg WC$", and so in fact $M \models$ "$\mathbf{V}_\kappa \models \mathsf{ZFC}+\mathsf{A}$". Hence $M \models$ "$\mathbf{V}_\kappa \models \text{Cons}(\mathsf{ZFC}+\varphi)$", and since this is an arithmetical (so Δ_0) statement $M \models \text{Cons}(\mathsf{ZFC} + \varphi)$. It follows by completeness that $\mathsf{ZFC} \vdash \text{Cons}(\mathsf{ZFC} + \varphi)$, and so $\mathsf{ZFC} + \mathsf{A} \equiv_{\text{Cons}} \mathsf{ZFC}$.

Now assuming that $\mathsf{ZFC} + \mathsf{WC}$ is consistent then $\mathsf{ZFC} \not\vdash \mathsf{A}$. Indeed, if $\mathsf{ZFC} \vdash \mathsf{A}$ then $\mathsf{ZFC} + \mathsf{WC} \vdash \mathsf{A}$ so if κ is the least weakly compact, find $\lambda > \kappa$ inaccessible. Then $\mathbf{V}_\lambda \models$ "κ is weakly compact" and so in fact $\mathbf{V}_\lambda \models \mathsf{ZFC} + \mathsf{WC}$ by inaccessibility. Hence $\mathsf{ZFC} + \mathsf{WC} \vdash \text{Cons}(\mathsf{ZFC} + \mathsf{WC})$, which is a contradiction on the assumption of the consistency of $\mathsf{ZFC} + \mathsf{WC}$ $\dashv$

(36) Suppose that there are unboundedly many inaccessible cardinals. Let ι_α be the αth inaccessible cardinal. Show that it is not possible to prove (in ZFC+"there are unboundedly many inaccessible cardinals") that the operation $\alpha \mapsto \iota_\alpha$ has a fixed point, i.e., some $\kappa = \iota_\kappa$. This must mean that the operation is in general not a normal ordinal operation. What is the reason?

Solution. Write $u\mathsf{IC}$ for the statement that there exist unboundedly many inaccessible cardinals, and suppose that we could prove in $\mathsf{ZFC} + u\mathsf{IC}$ that the above operation has a fixed point. Let $\kappa = \iota_\kappa$ be the least fixed point. We argue that $\mathbf{V}_\kappa \models \mathsf{ZFC} + u\mathsf{IC}$. Clearly, $\mathbf{V}_\kappa \models \mathsf{ZFC}$ by inaccessibility of κ. Furthermore, let $\alpha \in \text{Ord} \cap \mathbf{V}_\kappa = \kappa$. Since $\alpha < \kappa$, it follows that $\alpha \neq \iota_\alpha$, and so in particular $\alpha < \iota_\alpha < \iota_\kappa = \kappa$. Hence, $\iota_\alpha \in \mathbf{V}_\kappa$ and $\mathbf{V}_\kappa \models$ "ι_α is inaccessible" by ES1. It follows that $\mathbf{V}_\kappa \models \forall x(x \in \text{Ord} \to \exists y(x < y \wedge y \text{ is inaccessible}))$, thus $\mathbf{V}_\kappa \models u\mathsf{IC}$. Hence, $\mathsf{ZFC} + u\mathsf{IC} \vdash \text{Cons}(\mathsf{ZFC} + u\mathsf{IC})$, contradiction. This implies that we cannot show in $\mathsf{ZFC} + u\mathsf{IC}$ that this is a normal ordinal operation. In fact, we can prove that it is not. Clearly $\alpha < \beta \to \iota_\alpha < \iota_\beta$, so it must be that $\iota_\lambda \neq \bigcup_{\alpha<\lambda} \iota_\alpha$ for at least one limit ordinal λ. Indeed, for $\lambda = \omega$ we see that $\text{cf}(\bigcup_{n<\omega} \iota_n) = \text{cf}(\omega) = \omega$ which is clearly not the ω-th inaccessible. $\dashv$

(37) Show that if U is an ultrafilter, then U is free if and only if U is non-trivial.

Solution. Let U be an ultrafilter on a set I, and suppose that U is trivial, i.e. it contains a singleton $\{x\}$. Since $A \cap B \neq \emptyset$ for any $A, B \in U$, it must be that $x \in A$ for all $A \in U$. Hence, $\bigcap U = \{x\} \neq \emptyset$, i.e. U is fixed.

Conversely, suppose that U is non-trivial. Since no singleton $\{x\}$ is in U and U is an ultrafilter, it must be that $I \setminus \{x\} \in U$ for all $x \in I$. Since $\bigcap_{x\in I}(I \setminus \{x\}) = \emptyset$, it follows that $\bigcap U = \emptyset$. $\dashv$

(38) Presentation Example. Let λ be inaccessible and $M \subseteq \mathbf{V}_\lambda$ a transitive set. Suppose $j : \mathbf{V}_\lambda \to M$ is an elementary embedding. Show that if $j \neq \text{id}$, then there is an ordinal α such that $j(\alpha) > \alpha$.

Solution. Let $j : \mathbf{V}_\lambda \to M$ be a non-trivial elementary embedding with $M \subseteq \mathbf{V}_\lambda$ transitive. Suppose for a contradiction that j is the identity on ordinals of $\mathbf{V}_\lambda$. Then for all $x \in \mathbf{V}_\lambda$ $\text{rank}(j(x)) = j(\text{rank}(x))$ by absoluteness of rank, and that is equal to $\text{rank}(x)$ by the assumption on j. So let $x \in \mathbf{V}_\lambda$ be of least rank such that $x \neq j(x)$. It follows that for all $y \in j(x)$, $\text{rank}(y) < \text{rank}(j(x)) = \text{rank}(x)$ so $y = j(y)$. Hence $y \in j(x) \iff j(y) \in j(x) \iff y \in x$, and therefore $x = j(x)$, contradiction.

Finally, let α be least with $j(\alpha) \neq \alpha$. Then, for all $\beta \in \alpha$, $\beta = j(\beta) \in j(\alpha)$. Hence $\alpha < j(\alpha)$. $\dashv$

(39) We assume that $\kappa < \lambda$ are measurable and inaccessible, respectively, and that $j : \mathbf{V}_\lambda \to M$ is the ultrapower embedding. We use the notation from the lectures. In Lecture XI, we showed that $\kappa \leq (\text{id}) < j(\kappa)$. Give concrete functions $f : \kappa \to \kappa$ such that $(f) = (\text{id}) + 1$, $(f) = (\text{id}) + \omega_1$, $(f) = (\text{id}) \cdot 2$. Fix $\xi < \kappa$ and consider the function $f(\alpha) := \xi$ if α is even and $f(\alpha) := \alpha$ if α is odd. What can we say about the relation between (id) and (f)?

[*As usual, an ordinal α is even if it is of the form $\lambda + 2n$ where λ is a limit ordinal and n is a natural number.*]

Solution. Let $f_1 : \kappa \to \kappa$ be given by $\alpha \mapsto \alpha + 1$. Clearly (f_1) is an ordinal, and furthermore $\{\alpha < \kappa : f(\alpha) = \text{id}(\alpha) + 1\} \in U$, so $(f_1) = (\text{id}) + j(1) = (\text{id}) + 1$. Similarly, taking $f_2 : \alpha \mapsto \alpha + \omega_1$ and $f_3 : \alpha \mapsto \alpha \cdot 2$ we may show that $(f_2) = (\text{id}) + \omega_1$ while $(f_3) = (\text{id}) \cdot 2$.

Now fix $\xi < \kappa$ and take $f : \kappa \to \kappa$ with $f(\alpha) := \xi$ if α is even and $f(\alpha) := \alpha$ if α is odd. Let $E := \{\alpha < \kappa : \alpha \text{ is even}\} \subseteq \kappa$, and $O := \kappa \setminus$. By regularity of κ we can see that both E and O have size κ, so either one could be in U. If $E \in U$ then $(f) = j(\xi) = \xi < (\text{id})$, while if $O \in U$ then $(f) = (\text{id})$. In either case, $(f) \leq (\text{id})$. $\dashv$

(40) Let κ be measurable. Show that there is some ultrafilter U on κ such that in the ultrapower M_U, we have that $\kappa = (\text{id})_U$ where $\text{id} : \kappa \to \kappa : \alpha \mapsto \alpha$.

Solution. Let κ be measurable, and U a non-principal κ-complete ultrafilter on κ. Fix $f : \kappa \to \kappa$ so that $(f)_U = \kappa$, and let $W = \{X \subseteq \kappa : f^{-1}[X] \in U\}$. We argue that this is a non-principal κ-complete ultrafilter, and furthermore that $\kappa = (\text{id})_W$. Indeed this is clearly a filter, while if $X \notin W$ then $f^{-1}[X] \notin U$, hence $\kappa \setminus f^{-1}[X] = f^{-1}[\kappa \setminus X] \in U$, i.e. $\kappa \setminus X \in W$. Furthermore, this is non-principal. Indeed, if $\{\gamma\} \in W$ then $f^{-1}[\{\gamma\}] \in U$, i.e $\{\alpha < \kappa : f(\alpha) = \gamma\} \in U$, and so $(f)_U = j(\gamma) = \gamma$, contradiction. Finally, this is κ-complete as $\bigcap_{\gamma<\alpha} f^{-1}[X_\gamma] = f^{-1}[\bigcap_{\gamma<\alpha} X_\gamma]$ for any $\alpha < \kappa$.

It remains to show that $(\text{id})_W = \kappa$. We already know that $\kappa \leq (\text{id})_W$, so let $g : \kappa \to \kappa$ be such that $(g)_W \in (\text{id})_W$. It follows that $S = \{\alpha < \kappa : g(\alpha) < \alpha\} \in W$, so $f^{-1}[S] = \{\beta < \kappa : g(f(\beta)) < f(\beta)\} \in U$, and therefore $(g \circ f)_U < (f)_U$. So, there exists $\gamma < \kappa$ such that $(g \circ f)_U = \gamma = j(\gamma)$, i.e. $\{\beta < \kappa : g(f(\beta)) = \gamma\} \in U$. It follows that $\{\alpha < \kappa : g(\alpha) = \gamma\} \in W$, i.e. $(g)_W = \gamma < \kappa$. Hence $(\text{id})_W \subseteq \kappa$, implying that $(\text{id})_W = \kappa$. $\dashv$

Remark. A κ-complete non-principal ultrafilter on κ with the property that $\kappa = (\text{id})_U$ is called *normal*. The above therefore shows that every measurable cardinal has a normal ultrafilter.

(41) Let κ be a cardinal. We say κ is

0-inaccessible if κ is inaccessible

$\alpha + 1$-*inaccessible* if κ is α-inaccessible and $\{\mu < \kappa\,;\, \mu \text{ is } \alpha\text{-inaccessible}\}$ is unbounded in κ, and

λ-*inaccessible* if κ is α-inaccessible for all $\alpha < \lambda$ and λ is a limit ordinal.

Show that every measurable cardinal κ is κ-inaccessible.

Solution. Let κ be measurable and $j : \mathbf{V}_\lambda \to M$ be the elementary embedding defined from it. We saw in the lectures that for $\gamma \leq \kappa$, the statement "γ is inaccessible" is absolute between $\mathbf{V}_\lambda$ and M (Lecture XII, page 6) and that this implies by the technique of reflection that κ is 1-inaccessible (Lecture XII, page 7).

It is easy to see by induction that for $\gamma \leq \kappa$ and any α, the statement "γ is α-inaccessible" is absolute between $\mathbf{V}_\lambda$ and M.

Now we can prove by induction on $\alpha < \kappa$ that κ is α-inaccessible: as mentioned, the case $\alpha = 1$ was already proved in the lectures.

If λ is a limit ordinal and κ is α-inaccessible for all $\alpha < \lambda$, then κ is λ-inaccessible by definition.

If κ is α-inaccessible in $\mathbf{V}_\lambda$, then by absoluteness, it is α-inaccessible in M. Thus, for each $\gamma < \kappa$, M is a model of "there is some mu such that $j(\gamma) < \mu < j(\kappa)$ which is $j(\alpha)$-inaccessible". By elementarity, $\mathbf{V}_\lambda$ is a model of "there is some mu such that $\gamma < \mu < \kappa$ which is α-inaccessible". Since γ was arbitrary, this means that κ is $(\alpha + 1)$-inaccessible in $\mathbf{V}_\lambda$. So, by definition, κ is κ-inaccessible.

$\dashv$

(42) Let λ be inaccessible. Suppose that $M \subseteq \mathbf{V}_\lambda$ is an inner model of ZFC closed under κ-sequences (i.e., $M^\kappa \subseteq M$) with $\mathbf{V}_{\kappa+1} \subseteq M$, L is a language with at most κ many non-logical symbols, and that N is an L-structure with $|N| \leq \kappa$. Show that there is some $\overline{N} \in M$ such that N and $\overline{N}$ are isomorphic. Use this and (32) to finish the proof started in Lecture XIII that a measurable cardinal κ remains weakly compact in the ultrapower.

Solution. Since $|N| \leq \kappa$, w.l.g assume that there is $f : \kappa \to N$ which is a bijection. Let S be a non-logical symbol of L and consider the interpretation S^N of S in N. Then we may interpret S in κ through f, depending on whether S is a function or relation symbol. For example, if S is an α-ary relation symbol for $\alpha < \kappa$, then let $S^{\underline{\kappa}} := \{\vec{x} \in \kappa^\alpha : f(\bar{x}) \in S^N\}$. Since $\kappa \in M$ and M is closed under κ-sequences, it follows that $S^{\underline{\kappa}} \in M$. We proceed by interpreting all κ non-logical symbols in this way, and write $\underline{\kappa}$ for the resulting L-structure. Again, since M is closed under κ-sequences it follows that $\underline{\kappa} \in M$, while the function $f : \kappa \to N$ is an isomorphism of L-structures by construction.

We can now prove that every measurable cardinal κ remains weakly compact in the ultrapower. Indeed, let L be an $\mathcal{L}_{\kappa,\kappa}$ language with at most κ many non-logical symbols, and Φ a set of L-sentences such that $M \models$ "Φ is κ-satisfiable". As in Lecture XIII, this implies that $\mathbf{V}_\lambda \models$ "Φ is κ-satisfiable", and since κ is weakly compact in $\mathbf{V}_\lambda$ we have that $\mathbf{V}_\lambda \models$ "Φ is satisfiable". Hence, there is some $N \in \mathbf{V}_\lambda$ such that $\mathbf{V}_\lambda \models$ "$N \models \Phi''$, i.e. $N \models \Phi$. Starting from any substructure X of N of size $\leq \kappa$, we may construct an L-elementary substructure $H_\kappa(X) \preccurlyeq N$ of size $\leq \kappa$ using (32). By the previous argument, we may find some $\overline{N} \in M$ such that $H_\kappa(X)$ and $\overline{N}$ are isomorphic. It follows that $\overline{N} \models \Phi$, and so $M \models$ "$\overline{N} \models \Phi''$, i.e. $M \models$ "$\overline{N}$ is consistent". Hence $M \models$ "κ is weakly compact". $\dashv$

(43) Presentation Example. In Lecture XIV (page 6), we showed that if κ is surviving, there are functions f and g such that

$$M_U \models (g)_U \text{ is an } (f)_U\text{-complete ultrafilter on } (f)_U.$$

Use this to give an alternative proof of the fact that a surviving cardinal κ must be the κth measurable cardinal.

Solution. Let κ be a surviving cardinal, with U, V κ-complete non-principal ultrafilter on κ such that $V = (g)_U \in M_U$, and $\kappa = (f)_U$. Since $M_U \models (g)_U$ is an $(f)_U$-complete ultrafilter on $(f)_U$ it follows by Łoś's that $\{\alpha < \kappa : g(\alpha) \text{ is an } f(\alpha)\text{-complete ultrafilter on } f(\alpha)\} \in U$. Fix some $\gamma < \kappa$. Since $\gamma = j(\gamma)$ it follows that $\{\alpha < \kappa : \gamma < f(\alpha)\} \in U$. Hence, we can find some $\alpha < \kappa$ such that $f(\alpha) > \gamma$ and $g(\alpha)$ is an $f(\alpha)$-complete ultrafilter on $f(\alpha)$. Hence $f(\alpha)$ is measurable, and since $f(\alpha) < \kappa$, the set of measurables below κ is unbounded in κ. It follows that κ is the κ-th measurable. $\dashv$

(44) Show that if κ is 2-strong and satisfies $o(\kappa) \geq n$, then there are unboundedly many cardinals $\lambda < \kappa$ such that $o(\lambda) \geq n$.

Solution. Suppose that κ is 2-strong with Mitchell rank $\geq n$, i.e. there are $(U_i)_{i \in n}$ κ-complete non-principal ultrafilters on κ with $U_i < U_j \iff i < j$. Since κ is 2-strong there is an inner model M and an elementary embedding $j : \mathbf{V}_\lambda \to M$ such that $\mathbf{V}_{\kappa+2} \subseteq M$. But $U_1, \ldots, U_n \in \mathbf{V}_{\kappa+2}$, so $U_1, \ldots, U_n \in M$. Furthermore, it follows from the lectures (Lecture XIV, page 7) that $M \models U_i < U_j$ if and only if $i < j$. Hence for any $\alpha < \kappa$, $M \models \alpha < \kappa < j(\kappa) \wedge o(\kappa) \geq n$ and so $M \models \exists \beta(\alpha < \beta < j(\kappa) \wedge o(\beta) \geq n)$. Finally, since $j(\alpha) = \alpha$ we may use elementarity to obtain that $\mathbf{V}_\lambda \models \exists \beta(\alpha < \beta < \kappa \wedge o(\beta) \geq n)$, and hence there is some $\beta > \alpha$ with $o(\beta) \geq n$. It follows that there are unboundedly many $\lambda < \kappa$ such that $o(\lambda) \geq n$. $\dashv$

(45) Let κ be measurable and M the ultrapower built from a κ-complete ultrafilter on κ. Show that M is not closed under κ^+-sequences by producing a function $f : \kappa^+ \to M$ that is not an element of M.

Solution. Consider the map $f : \kappa^+ \to M$ given by $\alpha \mapsto j(\alpha)$. Then by Replacement in M, the set $j[\kappa^+] = \{j(\alpha) : \alpha < \kappa^+\}$ is in M. However $\bigcup j[\kappa^+] = j(\kappa^+)$. Indeed, if $(f)_U \in j(\kappa^+)$ then w.l.g $\mathrm{Im}(f) \subseteq \kappa^+$ and so $\exists \alpha < \kappa^+$ such that $\mathrm{Im}(f) \subseteq \alpha$ by regularity of κ^+. It follows that $(f)_U \leq j(\alpha)$, so $(f)_U \in \bigcup j[\kappa^+]$. But then $M \models$ "$|j[\kappa^+]| = \kappa^+$" as witnessed by $f \in M$, and furthermore $M \models$ "$j(\kappa^+)$ is regular" $\wedge$"$|j(\kappa^+)| \geq (2^\kappa)^+$", contradiction. $\dashv$