Example Sheet #3

Examples Classes.
#1: Friday 11 February 2022, 3:30–5pm, **MR4**.
#2: Friday 25 February 2022, 3:30–5pm, **online on Zoom**.
#3: Friday 18 March 2022, 3:30–5pm, **MR4**.

Presentation. Two of the examples are designed to be a Presentation Example (marked on the sheet). We encourage all students to meet in pairs, work together on these examples, and prepare a short presentation of their solutions that can be given on the blackboard in **MR4** during the examples class. The discussion during your meeting should be both about the mathematical content and about the preparation of the presentation.

Marking. You can submit all of your work to Ioannis Eleftheriadis (`ie257`) as a *single pdf file* by e-mail or hand it to him on paper during the examples class. Please submit all work before the start of the examples class. Work that is submitted at least 24 hours before the examples class could already be marked and returned during the examples class. We cannot guarantee that all work will be marked, but we shall endeavour to mark at least two examples per submission. Model solutions will be provided on the moodle page of the course.

(31) Let κ be inaccessible and L be any $\mathcal{L}_{\kappa\kappa}$ language and M an L-structure. Write L^α for the set of L-formulas whose free variables are contained in $\{v_\xi\,;\,\xi<\alpha\}$. If $X\subseteq M$, we say that X is an *L-elementary substructure* (in symbols: $X\preccurlyeq_L M$) if for all $\varphi\in L^\alpha$ and all $\vec{x}\in X^\alpha$, we have that

$$X\frac{\vec{x}}{\vec{v}}\models\varphi \iff M\frac{\vec{x}}{\vec{v}}\models\varphi.$$

Prove the following statement (*Tarski-Vaught Test for $\mathcal{L}_{\kappa\kappa}$ languages*): a subset X is an L-elementary substructure if and only if it is an L-substructure and for all $\varphi\in L^{\alpha+\beta}$ (with $\vec{v}:=\{v_\xi\,;\,\xi<\alpha\}$ and $\vec{w}:=\{v_{\alpha+\eta}\,;\,\eta<\beta\}$) and all $\vec{x}\in X^\alpha$, if $M\frac{\vec{x}}{\vec{v}}\models\exists^\beta\vec{w}\varphi$, then there is some $\vec{y}\in X^\beta$ such that $M\frac{\vec{x}}{\vec{v}}\frac{\vec{y}}{\vec{w}}\models\varphi$. (Why do we require the inaccessibility of κ?)

(32) Let κ be inaccessible, L be any $\mathcal{L}_{\kappa\kappa}$ language, M an L-structure, and $X\subseteq M$. If $\varphi\in L^{\alpha+\beta}$ (with $\vec{v}:=\{v_\xi\,;\,\xi<\alpha\}$ and $\vec{w}:=\{v_{\alpha+\eta}\,;\,\eta<\beta\}$) and and $\vec{x}\in M^\alpha$ such that $M\frac{\vec{x}}{\vec{v}}\models\exists^\beta\vec{w}\varphi$, then there is some $\vec{y}\in M^\beta$ such that $M\frac{\vec{x}}{\vec{v}}\frac{\vec{y}}{\vec{w}}\models\varphi$. Use the Axiom of Choice to assign such a witness $\mathrm{w}(\varphi,\vec{x})$. Let $H(X,\alpha):=X\cup\bigcup\{\mathrm{ran}(\mathrm{w}(\varphi,\vec{x}))\,;\,\varphi\in L^{\alpha+\alpha},\ \vec{x}\in X^\alpha\}$. Define by recursion $H_0(X):=X$, $H_{\alpha+1}(X):=H(H_\alpha(X,\alpha))$, and $H_\lambda(X):=\bigcup_{\alpha<\lambda}H_\alpha(X)$ (for limit ordinals λ) and show that $H_\kappa(X)\preccurlyeq_L M$ is an elementary substructure of cardinality $\leq\kappa$.

(33) Show that the consistency strength hierarchy has the following properties:

(a) $0=1$ is maximal w.r.t. $\leq_{\mathrm{Cons}}$;

(b) if A is not maximal, then there is B such that $\mathsf{A}<_{\mathrm{Cons}}\mathsf{B}$ and B is not maximal;

(c) for all A and B, if $\mathsf{A}\leq_{\mathrm{Cons}}\mathsf{B}$, then $\mathsf{A}\vee\mathsf{B}\equiv_{\mathrm{Cons}}\mathsf{A}$.

(34) Let Φ be a cardinal property (i.e., $\Phi(\kappa)$ implies that κ is a cardinal). Let us say that Φ is *nontrivial* if $\Phi(\kappa)$ implies that κ is inaccessible. Show that there is a nontrivial Φ such that $\Phi\mathsf{C} \equiv_{\text{Cons}} \mathsf{IC}$ and $\mathsf{WC} <_1 \Phi\mathsf{C}$. Use this to argue that the following statement is in general false: if $\mathsf{A} \leq_{\text{Cons}} \mathsf{B}$, then $\mathsf{A} \wedge \mathsf{B} \equiv_{\text{Cons}} \mathsf{B}$.

(35) Let A be the statement "if there is a weakly compact cardinal κ, then there is an inaccessible $\lambda > \kappa$". Show that the consistency strength of $\mathsf{ZFC}+\mathsf{A}$ is equal to that of ZFC, but that under some consistency assumptions, $\mathsf{ZFC} <_0 \mathsf{ZFC}+\mathsf{A}$. What are the required consistency assumptions for the latter claim?

(36) Suppose that there are unboundedly many inaccessible cardinals. Let ι_α be the αth inaccessible cardinal. Show that it is not possible to prove (in ZFC+"there are unboundedly many inaccessible cardinals") that the operation $\alpha \mapsto \iota_\alpha$ has a fixed point, i.e., some $\kappa = \iota_\kappa$. This must mean that the operation is in general not a normal ordinal operation. What is the reason?

(37) Show that if U is an ultrafilter, then U is free if and only if U is non-trivial.

(38) Presentation Example. Let λ be inaccessible and $M \subseteq \mathbf{V}_\lambda$ a transitive set. Suppose $j : \mathbf{V}_\lambda \to M$ is an elementary embedding. Show that if $j \neq \text{id}$, then there is an ordinal α such that $j(\alpha) > \alpha$.

(39) We assume that $\kappa < \lambda$ are measurable and inaccessible, respectively, and that $j : \mathbf{V}_\lambda \to M$ is the ultrapower embedding. We use the notation from the lectures. In Lecture XI, we showed that $\kappa \leq (\text{id}) < j(\kappa)$. Give concrete functions $f : \kappa \to \kappa$ such that $(f) = (\text{id}) + 1$, $(f) = (\text{id}) + \omega_1$, $(f) = (\text{id}) \cdot 2$. Fix $\xi < \kappa$ and consider the function $f(\alpha) := \xi$ if α is even and $f(\alpha) := \alpha$ if α is odd. What can we say about the relation between (id) and (f)?

[*As usual, an ordinal α is even if it is of the form $\lambda + 2n$ where λ is a limit ordinal and n is a natural number.*]

(40) Let κ be measurable. Show that there is some ultrafilter U on κ such that in the ultrapower M_U, we have that $\kappa = (\text{id})_U$ where $\text{id} : \kappa \to \kappa : \alpha \mapsto \alpha$.

(41) Let κ be a cardinal. We say κ is

0-inaccessible if κ is inaccessible

$\alpha + 1$-*inaccessible* if κ is α-inaccessible and $\{\mu < \kappa\,;\, \mu \text{ is } \alpha\text{-inaccessible}\}$ is unbounded in κ, and

λ-*inaccessible* if κ is α-inaccessible for all $\alpha < \lambda$ and λ is a limit ordinal.

Show that every measurable cardinal κ is κ-inaccessible.

(42) Let λ be inaccessible. Suppose that $M \subseteq \mathbf{V}_\lambda$ is an inner model of ZFC closed under κ-sequences (i.e., $M^\kappa \subseteq M$) with $\mathbf{V}_{\kappa+1} \subseteq M$, L is a language with at most κ many non-logical symbols, and that N is an L-structure with $|N| \leq \kappa$. Show that there is some $\overline{N} \in M$ such that N and $\overline{N}$ are isomorphic. Use this and (32) to finish the proof started in Lecture XIII that a measurable cardinal κ remains weakly compact in the ultrapower.

(43) Presentation Example. In Lecture XIV (page 6), we showed that if κ is surviving, there are functions f and g such that

$$M_U \models (g)_U \text{ is an } (f)_U\text{-complete ultrafilter on } (f)_U.$$

Use this to give an alternative proof of the fact that a surviving cardinal κ must be the κth measurable cardinal.

(44) Show that if κ is 2-strong and satisfies $o(\kappa) \geq n$, then there are unboundedly many cardinals $\lambda < \kappa$ such that $o(\lambda) \geq n$.

(45) Let κ be measurable and M the ultrapower built from a κ-complete ultrafilter on κ. Show that M is not closed under κ^+-sequences by producing a function $f : \kappa^+ \to M$ that is not an element of M.