Model Solutions for Example Sheet #2

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(15) Modify the proof that ZFC (if consistent) does not prove IC (Lecture II, page 4) to a proof of "if ZFC + GCH is consistent, then ZFC does not prove that there are weakly inaccessible cardinals". Argue that this gives rise to a proof of the unprovability of the existence of weakly inaccessibles that does not need all of Gödel's 1938 theorem (Lecture V, page 6).

Solution. Suppose that $\mathsf{ZFC} \vdash$ WIC. Since $\mathsf{ZFC} + \mathsf{GCH} \vdash \forall\kappa(\kappa$ is weakly inaccessible $\iff$ κ is inaccessible), this implies that $\mathsf{ZFC}+\mathsf{GCH} \vdash$ IC. But $\mathsf{ZFC}+$IC $\vdash$ Cons(ZFC), so $\mathsf{ZFC}+\mathsf{GCH} \vdash$ Cons(ZFC). But by Gödel's theorem, Cons(ZFC) $\to$ Cons(ZFC + GCH), and so $\mathsf{ZFC} + \mathsf{GCH} \vdash$ Cons(ZFC + GCH), contradiction. Technically this does not use the full strength of Gödel's theorem: Gödel built an $\emptyset$-definable inner model of ZFC + GCH within ZF, but the implication Cons(ZFC) $\to$ Cons(ZFC + GCH) requires any inner model of ZFC + GCH built within ZFC. ⊣

(16) Let 2IC be the statement "there are $\lambda < \kappa$ such that both λ and κ are inaccessible". Show that if ZFC + IC is consistent, then IC does not imply 2IC.

Solution. Suppose that there is an inaccessible cardinal. Let λ be the least inaccessible cardinal, and κ the next inaccessible. Consider $\mathbf{V}_\kappa$. Since $\lambda < \kappa$ is inaccessible $\mathbf{V}_\kappa \models$ IC. However $\mathbf{V}_\kappa \not\models$ 2IC, as there is only one inaccessible below κ. So IC cannot possibly imply 2IC. ⊣

(17) Show that there is a Π_1 formula ϕ such that ZFC proves $\phi(x)$ iff x is a strong limit cardinal.

Solution. By the previous sheet we may assume the existence of a Π_1 formula card(x) which is provable in ZFC if and only if x is a cardinal. Then

$$\mathrm{card}(x) \wedge \forall(y \in x)\exists(z \in x)(\mathrm{card}(z) \wedge z > 2^y), \text{ where}$$

$$(z > 2^y) := \forall f(\mathrm{function}(f, \mathcal{P}(y), z) \to \neg\mathrm{surjection}(f, \mathcal{P}(y), z))$$

is (equivalent to) a Π_1 formula that expresses that x is a strong limit cardinal modulo ZFC. ⊣

(18) Remind yourself of Mostowski's Collapsing Theorem (Theorem 4 in § 5 of Imre Leader's notes for the course *Logic & Set Theory*). Let κ be inaccessible. In Lecture V, we constructed a countable, non-transitive $M \subseteq \mathbf{V}_\kappa$ such that $M \preccurlyeq \mathbf{V}_\kappa$. Use Mostowski's Collapsing Theorem to show that there is a transitive set $M^* \in \mathbf{V}_\kappa$ such that $(M^*, \in)$ is isomorphic to $(M, \in)$. In particular, $M^* \subseteq \mathbf{V}_\kappa$ is a transitive submodel of ZFC.

Solution. We ought to show that $\in$ is set-like, well-founded, and extensional on M. Clearly, $\in$ is set-like. Furthermore, if $S \subseteq M \subseteq \mathbf{V}_\kappa$ is non-empty, then it has an $\in$-minimal element in $\mathbf{V}_\kappa$ by the well-foundedness of $(\mathbf{V}_\kappa, \in)$. This will also be minimal in M. Finally, extensionality also follows trivially: if $x = y$ then $\{z \in M : z \in x\} = \{z \in M : z \in y\}$. Now $V_\kappa \models$ ZFC, so in particular it proves the Mostowski collapse lemma. So $\mathbf{V}_\kappa \models$ "$(M, \in)$ is order-isomorphic to a unique transitive $(M^*, \in)$". Being an isomorphism and a transitive set are both absolute between transitive classes, so $(M^*, \in)$ is indeed a transitive set in $\mathbf{V}_\kappa$ that models ZFC and is isomorphic to $(M, \in)$. ⊣

(19) Using the model M^* from 18 explain why Π_1 formulas are not in general absolute between transitive models of ZFC.

[*Hint.* What is $\mathrm{Ord} \cap M^*$? If $\kappa \in M^*$ is such that $M^* \models$ "κ is a cardinal", can κ be a real cardinal?]

Solution. Consider the Π_1 formula $\phi(x)$ saying that x is an uncountable ordinal. Since $M^* \models \mathsf{ZFC}$, we may find some $\alpha \in M^*$ such that $M^* \models \phi(\alpha)$. However it is clear that α is not really uncountable: $\alpha \in M^*$ implies $\alpha \subseteq M^*$ by transitivity, and M^* is countable. So $\mathbf{V} \not\models \phi(\alpha)$, and hence $\phi(x)$ is not absolute between M^* and $\mathbf{V}$. ⊣

(20) Presentation Example. Show that the smallest Ulam cardinal is a measurable cardinal.

Solution. Let κ be the least Ulam cardinal, and let U be a σ-complete non-principal ultrafilter on κ. Suppose that U is not κ-complete. We may therefore find a partition $\{X_\alpha : \alpha < \gamma\}$ of κ with $\gamma < \kappa$, such that $X_\alpha \notin U$ for all $\alpha < \gamma$. Define a surjection $f : \kappa \to \gamma$ by $f(x) = \alpha$ if and only if $x \in X_\alpha$. This induces a "pushforward" ultrafilter F on γ given by $Z \in F$ if and only if $f^{-1}(Z) \in U$. This is trivially σ-complete as $f^{-1}(\bigcap_{i\in\omega} S_i) = \bigcap_{i\in\omega} f^{-1}(S_i)$, while it is also non-principal. Indeed, if $\{\alpha\} \in F$ for some $\alpha < \gamma$ then $f^{-1}(\alpha) = X_\alpha \in U$, contradiction. Hence $\gamma < \kappa$ is Ulam, contradicting minimality. It follows that U is κ-complete, and so κ is measurable. ⊣

(21) Suppose $\mu : \kappa \to 2$ and $U \subseteq \wp(\kappa)$; define $\mu_U(A) := 1$ if $A \in U$ and $U_\mu := \{A\,;\, \mu(A) = 1\}$. Show that if U is a κ-complete nontrivial ultrafilter on κ, then μ_U is a κ-additive nontrivial measure on κ and if μ is a κ-additive nontrivial measure on κ, then U_μ is a κ-complete nontrivial ultrafilter on κ.

Solution. It is easy to see that the non-triviality of the measure is equivalent to the non-principality of the ultrafilter, so we only need to consider additivity and completeness. Suppose that U is κ-complete. Let $(X_\alpha)_{\alpha<\gamma}$ be a collection of disjoint subsets of κ with $\gamma < \kappa$. Suppose that $\bigcup_{\alpha<\gamma} X_\alpha \in U$. Then exactly one X_α must be in U: if none is then their union cannot be in U, while if more than one are then their intersection is $\emptyset$ and is in U, which is a contradiction. Furthermore, if $\bigcup_{\alpha<\gamma} X_\alpha \notin U$ then no X_α is in U by upwards closure. It follows that $\mu_U(\bigcup_{\alpha<\gamma} X_\alpha) = \sum_{\alpha<\gamma} \mu_U(X_\alpha)$.

Conversely, assume that μ is a κ-additive measure. Suppose for a contradiction that U_μ is not κ-complete. Find $\gamma < \kappa$ and a partition $(X_\alpha)_{\alpha<\gamma}$ of κ with $X_\alpha \notin U_\mu$ for all $\alpha < \gamma$. Then $\sum_{\alpha<\gamma} \mu(X_\alpha) = 0$, while $\mu(\bigcup_{\alpha<\gamma} X_\alpha) = \mu(\kappa) = 1$, contradicting κ-additivity. ⊣

(22) Let κ be regular. Show that $U = \{X \subseteq \kappa : |\kappa \backslash X| < \kappa\}$ is a κ-complete filter that is not an ultrafilter.

Solution. Clearly $\kappa \in U$, $\emptyset \notin U$. If $Y \supset X \in U$, then $|\kappa \setminus Y| \leq |\kappa \setminus X| < \kappa$ so $Y \in U$. If $X_1, X_2 \in U$, then $\kappa \setminus (X_1 \cap X_2) = (\kappa \setminus X_1) \cup (\kappa \setminus X_2)$, so $(X_1 \cap X_2) \in U$. Hence, this is a filter. If $(X_\alpha)_{\alpha<\gamma}$ is a collection of subsets of κ with $\gamma < \kappa$, then $\kappa \setminus (\bigcap_{\alpha<\gamma} X_\alpha) = \bigcup_{\alpha<\gamma}(\kappa \setminus X_\alpha)$. But $|\bigcup_{\alpha<\gamma}(\kappa \setminus X_\alpha)| \neq \kappa$ by regularity, so $\kappa \setminus (\bigcap_{\alpha<\gamma} X_\alpha) \in U$. It follows that U is κ-complete. Furthermore this is non-principal, as no singleton is in U. This implies that ZFC could not possibly prove that this is an ultrafilter. Indeed, if it did then ZFC would prove the existence of a measurable cardinal, and hence of an inaccessible cardinal. In fact, it would hold that every regular cardinal is measurable which of course is nonsense. More directly, we see that $X = \{\alpha < \kappa : \alpha \text{ is a limit ordinal or } 0\}$ and $Y = \{\alpha < \kappa : \alpha \text{ is a successor ordinal}\}$ are such that $X = \kappa \setminus Y$ and $|X| = |Y| = \kappa$ so none of these is in U, i.e. U is not an ultrafilter.

⊣

(23) Using the Axiom of Choice, show that every filter can be extended to an ultrafilter (preserving non-triviality).

Solution. Let F be a non-principal filter on a set X. Consider $S = \{U \subseteq \mathcal{P}(X) : U \text{ is a non-principal filter and } F \subseteq U\}$, which is a set by separation. Let $P = (X_i)_{i\in I}$ be a chain (w.r.t inclusion) in S, and consider $\bigcup P$. This is still a filter on X. Indeed $X \in \bigcup P$, while if $\emptyset \in \bigcup P$ then $\emptyset \in X_i$ for some i, contradiction. It is clearly upwards closed, while if $A, B \in \bigcup P$ then we may find some i such that $A, B \in X_i$ and so $A \cap B \in X_i \subseteq \bigcup P$. Since $\{a\} \in \bigcup P$ implies $\{a\} \in X_i$ for some i, this is a non-principal filter extending F so $\bigcup P \in S$. By Zorn's lemma, S therefore has a maximal element U. This is an ultrafilter: Suppose that $A \subseteq X$ is such that $A \notin U$ and $X \setminus A \notin U$, and take some $Y \in U$. If $Y \cap A = \emptyset$, then $Y \subseteq X$ so $X \setminus A \in U$, contradiction. So the set $\{Z \subseteq X : Z \supseteq Y \cap A \text{ or } Z \supseteq Y, Y \in U\}$

is an ultrafilter extending U, contradicting its maximality. Note that $U \in S$ so by definition it is non-principal. $\dashv$

Remark. Clearly this construction cannot preserve κ-completeness. Otherwise, the filter in Q? can be extended to a κ-complete non-principal ultrafilter and so ZFC would prove the existence of measurable cardinals. Essentially we end up putting so many subsets in F to make it an ultrafilter that completeness eventually breaks.

(24) A model $(M, E) \models \mathsf{ZFC}$ is called an ω-model if its natural numbers are standard, i.e., if there is an isomorphism between $(\{x \in M\,;\, M \models \text{“}x \text{ is a natural number”}\}, E)$ and $(\omega, \in)$. Let M be an ω-model; without loss of generality, we can assume that $\omega \subseteq M$. We encode formulas of first-order logic by natural numbers, writing $\ulcorner\varphi\urcorner$ for the number coding φ. Let Φ be a set of first-order sentences such that Φ exists in M, i.e., there is some $x \in M$ such that $\varphi \in \Phi$ if and only if $M \models \ulcorner\varphi\urcorner \in x$. Show that Φ is consistent if and only if $M \models$ “Φ is consistent”. Deduce that if $\mathsf{ZFC} + \mathrm{Cons}(\mathsf{ZFC})$ is consistent, it cannot show the existence of an ω-model.

Solution. Call a formula φ in the language of set theory *arithmetical* if there is a formula ψ in the language of arithmetic such that ZFC proves that φ is equivalent to “$\mathbb{N} \models \psi$” (the latter refers to the natural formula expressing this in the language of set theory). If M is an ω-model, then arithmetical formulae are absolute for M (via the isomorphism).

Suppose Φ is inconsistent. This means that there is some finite $\Phi_0 := \{\varphi_0, ..., \varphi_n\} \subseteq \Phi$ which is inconsistent. The statement “there is a proof of $0 = 1$ from $\varphi_0, ..., \varphi_n$” is an arithmetical statement, therefore absolute for M, so true in M. But then $M \models$“Φ is inconsistent”.

Suppose $M \models$“Φ is inconsistent”. This means that in M, there is a finite sequence of formulas that constitutes a proof of $0 = 1$ with assumptions from Φ. Since M is an ω-model, the length of this sequence is a natural number and there is a finite subset of $\Phi_0 := \{\varphi_0, ..., \varphi_n\} \subseteq \Phi$ such that this object is (the M-representation of) a Φ_0-proof of $0 = 1$. But now again, “there is a proof of $0 = 1$ from $\varphi_0, ..., \varphi_n$” is an arithmetical statement, therefore absolute for M, so Φ_0 is really inconsistent.

For the second part of the question, let Φ be a computable theory, i.e., a theory such that there is a fixed register machine R that enumerates all of its formulae. The machine R is encoded by a natural number, so M can refer to it. Therefore $\ulcorner\varphi\urcorner \in \Phi$ is described by an arithmetical formula (“there is a natural number n and a finite sequence of computational snapshots according to the machine R such that the sequence starts with input n and halts at the end with output $\ulcorner\varphi\urcorner$”) and thus absolute for M. Use Separation in M to obtain the set of all natural numbers satisfying that formula; by absoluteness, this describes Φ, so in the sense defined in the example statement, “Φ exists in M”.

Since ZFC is a computable theory, the above argument shows it “exists in M” and thus we can apply the first part to ZFC. Now assume towards a contradiction that $\Phi := \mathsf{ZFC} + \mathsf{Cons}(\mathsf{ZFC})$ implies the existence of an ω-model $M \models \mathsf{ZFC}$. Work in an arbitrary model of Φ. Since ZFC is computable, we can apply the first part and get that $M \models \mathrm{Cons}(\mathsf{ZFC})$, so $M \models \Phi$. But since we were working in an arbitrary model of Φ, this means that we proved $\mathrm{Cons}(\Phi)$ from Φ. By Gödel's Incompleteness Theorem, this implies that Φ is inconsistent. $\dashv$

(25) Find an $L_{\omega_1\omega}$ formula that characterises the ω-models of ZFC.

Solution. The ω-models of ZFC are precisely those models M of ZFC such that

$$M \models \forall x (x \in \omega \to \bigvee_{n \in \omega} x = \mathrm{succ}^n(\emptyset))$$

. $\dashv$

(26) Give a concrete uncountable collection of $\mathcal{L}_{\omega_1\omega}$ sentences that is countably satisfiable, but not satisfiable.

Solution. Consider the $\mathcal{L}_{\omega_1\omega}$ language with constant symbols $\{c_n, c_n^0, c_n^1 : n \in \omega\}$. Let S be the set of sentences

$$\{\bigwedge_{n\in\omega}(c_n^0 \neq c_n^1), \bigwedge_{n\in\omega}(c_n = c_n^0 \vee c_n = c_n^1)\} \cup \{\bigvee_{n\in\omega}(c_n \neq c_n^{f(n)}) : f : \omega \to 2\}$$

This is clearly countably satisfiable but not satisfiable. $\dashv$

(27) If κ is a strongly compact cardinal, the Keisler-Tarski theorem makes a statement about κ-complete filters on arbitrary sets X. What does the proof show if κ is only assumed to be weakly compact? Why is that useless?

[*Hint.* If $\lambda < \kappa$, which filters on λ can be κ-complete?]

Solution. The Keisler-Tarski theorem says that if κ is strongly compact, and F is a filter on an arbitrary set X, then this can be extended to a κ-complete ultrafilter on X. The proof of this argues by considering an $\mathcal{L}_{\kappa\kappa}$ language with $2^{|X|}$ non-logical symbols. If κ is only assumed to be weakly compact, then it must be that $2^{|X|} \leq \kappa$, and so $|X| < \kappa$ since it is a strong limit. However, if $\lambda < \kappa$ then all κ-complete ultrafilters on λ will be principal. Indeed, $\lambda = \bigcup_{\alpha<\lambda}\{\alpha\}$, and so by κ-completeness exactly one singleton $\{\alpha\}$ must be in the ultrafilter. All principal ultrafilter are trivially κ-complete for all cardinals κ, so this does not really tell us anything. $\dashv$

(28) In a reflection argument, we used Keisler's Theorem on the Extension Property to show that below each weakly compact cardinal is an inaccessible by reflecting the property "κ is inaccessible". Clearly, it cannot be possible to reflect the property "κ is weakly compact". Explain where the argument breaks down if you try to prove this.

Solution. The reflection of the property "κ is inaccessible" relied on the fact that this can be expressed by a Π_1 formula modulo ZFC, and is therefore downwards absolute. Under our definition of weak compactness, this is clearly not the case. One can show that being weakly compact can be expressed by a Π_2 formula modulo ZFC using an equivalent characterisation of weak compactness in terms of a Ramsey-like property. In fact, being weakly compact could not possibly be a downwards absolute property. If it was then taking κ to be the least weakly compact cardinal, we may use the Keisler Extension Property to find some transitive $(X, \in) \succcurlyeq (\mathbf{V}_\kappa, \in)$ such that $\kappa \in X$. Then $X \models \mathrm{WC}$, so $\mathbf{V}_\kappa \models \mathrm{WC}$. If $\lambda < \kappa$ witnesses this, then using the Ramsey-like characterisation of weak compactness one may show that λ is actually a weakly compact cardinal, contradiction. $\dashv$

(29) Let $\infty\mathsf{IC}$ be the statement "for all ordinals α, there is $\kappa > \alpha$ such that κ is inaccessible". Show that if κ is weakly compact, then $\mathbf{V}_\kappa \models \infty\mathsf{IC}$.

Solution. Firstly note that modifying the above argument by taking κ to be the $(n+1)$-th inaccessible cardinal, we see that $\mathbf{V}_\kappa \models n\mathsf{IC}$ and $\mathbf{V}_\kappa \not\models \infty\mathsf{IC}$ so $\mathsf{ZFC} + n\mathsf{IC}$ does not imply $\infty\mathsf{IC}$.

Now, let κ be weakly compact. We shall show that the set of inaccessibles below κ is unbounded in κ. Indeed, fix some $\alpha < \kappa$. Then using the Keisler Extension Property, find some transitive $(X, \in) \succcurlyeq (\mathbf{V}_\kappa, \in)$ such that $\kappa \in X$. Clearly $X \models \alpha < \kappa$ so $X \models \exists\lambda(\lambda$ is inaccessible and $\alpha < \lambda)$. By elementarity $\mathbf{V}_\kappa \models \exists\lambda(\lambda$ is inaccessible and $\alpha < \lambda)$, and so there is some $\lambda < \kappa$ which is actually an inaccessible and $\alpha < \lambda$. It follows that $\mathbf{V}_\kappa \models \infty\mathsf{IC}$. $\dashv$

(30) Presentation Example. Suppose that κ is a measurable cardinal and U is a κ-complete ultrafilter on κ, and $\pi : \mathbf{V}_\kappa \to \mathrm{Ult}(\mathbf{V}_\kappa, U)$ is the ultrapower embedding, i.e., $\pi(x) := [\mathrm{c}_x]_U$. By Łoś's Theorem, π is an elementary embedding. Show that $\{\pi(x)\,;\, x \in \mathbf{V}_\kappa\}$ is isomorphic to $\mathbf{V}_\kappa$ and transitive in $\mathrm{Ult}(\mathbf{V}_\kappa, U)$, i.e., if $z \in \pi(x)$, then there is $y \in \mathbf{V}_\kappa$ such that $z = \pi(y)$.

Conclude that the order type of the ordinals of $\mathrm{Ult}(\mathbf{V}_\kappa, U)$ is not equal to κ and that therefore $\mathrm{Ult}(\mathbf{V}_\kappa, U)$ is not isomorphic to $\mathbf{V}_\kappa$.

Solution. It is evident that the elementary embedding π is injective, and is therefore an isomorphism between $\mathbf{V}_\kappa$ and $\pi[\mathbf{V}\kappa]$. Furthermore, $\pi[\mathbf{V}\kappa]$ is transitive in $\mathrm{Ult}(\mathbf{V}_\kappa, U)$. Indeed, if $z \in \pi(x)$ then $\{\alpha < \kappa : z(\alpha) \in x\} \in U$. Since $x \in \mathbf{V}_\kappa$, $|x| = \lambda < \kappa$ so fix some enumeration $x = \{y_\beta : \beta < \lambda\}$. Consider the sets $X_\beta = \{\alpha < \kappa : z(\alpha) = y_\beta\}$. Since $\bigcup_{\beta<\lambda} X_\beta = \{\alpha < \kappa : z(\alpha) \in x\} \in U$, this implies by κ-completeness that $X_\beta \in U$ for some $\beta < \lambda$. Hence $z = \pi(y_\beta)$.

By transitivity, it is clear that $\mathbf{On}^{\pi[\mathbf{V}_\kappa]} \subseteq \mathbf{On}^{\mathrm{Ult}(\mathbf{V}_\kappa, U)}$. We now show that the order type of the ordinals of $\mathrm{Ult}(\mathbf{V}_\kappa, U)$ is not equal to κ by constructing a "non-standard" ordinal in the ultrapower. Consider the embedding $f : \kappa \hookrightarrow \mathbf{V}_\kappa$. Since $\mathbf{V}_\kappa \models f(\alpha) \in \mathbf{On}$ for all $\alpha < \kappa$, $\mathrm{Ult}(\mathbf{V}_\kappa, U) \models [f] \in \mathbf{On}$. Furthermore, for each $\beta < \kappa$, the set $\{\alpha < \kappa : f(\alpha) > \beta\}$ is in U as its complement has size $< \kappa$. Hence $\mathrm{Ult}(\mathbf{V}_\kappa, U) \models [f] > \pi(\beta)$ for each $\beta < \kappa$. Since π is an isomorphism $\mathbf{On}^{\pi[\mathbf{V}_\kappa]} = \{\pi(\alpha) : \alpha < \kappa\}$, while $[f] \in \mathbf{On}^{\mathrm{Ult}(\mathbf{V}_\kappa, U)} \setminus \mathbf{On}^{\pi[\mathbf{V}_\kappa]}$. Hence, $\mathrm{Ult}(\mathbf{V}_\kappa, U)$ is not isomorphic to $\pi[\mathbf{V}_\kappa]$, and is therefore not isomorphic to $\mathbf{V}_\kappa$.

$\dashv$