Large Cardinals
Lent Term 2022
Part III of the Mathematical Tripos
University of Cambridge
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Example Sheet #2

Examples Classes.
#1: Friday 11 February 2022, 3:30–5pm, **MR4**.
#2: Friday 25 February 2022, 3:30–5pm, **MR4**.
#3: Friday 18 March 2022, 3:30–5pm, **MR4**.

Presentation. Two of the examples are designed to be a Presentation Example (marked on the sheet). We encourage all students to meet in pairs, work together on these examples, and prepare a short presentation of their solutions that can be given on the blackboard in **MR4** during the examples class. The discussion during your meeting should be both about the mathematical content and about the preparation of the presentation.

Marking. You can submit all of your work to Ioannis Eleftheriadis (`ie257`) as a *single pdf file* by e-mail or hand it to him on paper during the examples class. Please submit all work before the start of the examples class. Work that is submitted at least 24 hours before the examples class could already be marked and returned during the examples class. We cannot guarantee that all work will be marked, but we shall endeavour to mark at least two examples per submission. Model solutions will be provided on the moodle page of the course.

(15) Modify the proof that ZFC (if consistent) does not prove IC (Lecture II, page 4) to a proof of "if ZFC + GCH is consistent, then ZFC does not prove that there are weakly inaccessible cardinals". Argue that this gives rise to a proof of the unprovability of the existence of weakly inaccessibles that does not need all of Gödel's 1938 theorem (Lecture V, page 6).

(16) Let 2IC be the statement "there are $\lambda < \kappa$ such that both λ and κ are inaccessible". Show that if ZFC + IC is consistent, then IC does not imply 2IC.

(17) Show that there is a Π_1 formula φ such that $\mathsf{ZFC} \vdash \varphi(x)$ if and only if x is a strong limit cardinal.

(18) Remind yourself of Mostowski's Collapsing Theorem (Theorem 4 in § 5 of Imre Leader's notes for the course *Logic & Set Theory*). Let κ be inaccessible. In Lecture V, we constructed a countable, non-transitive $M \subseteq \mathbf{V}_\kappa$ such that $M \preccurlyeq \mathbf{V}_\kappa$. Use Mostowski's Collapsing Theorem to show that there is a transitive set $M^* \in \mathbf{V}_\kappa$ such that $(M^*, \in)$ is isomorphic to $(M, \in)$. In particular, $M^* \subseteq \mathbf{V}_\kappa$ is a transitive submodel of ZFC.

(19) Using the model M^* from (17), explain why Π_1 formulas are not in general absolute between transitive models of ZFC.

[*Hint.* What is $\mathrm{Ord} \cap M^*$? If $\kappa \in M^*$ is such that $M^* \models$ "κ is a cardinal", can κ be a real cardinal?]

(20) Presentation Example. Show that the smallest Ulam cardinal is a measurable cardinal.

(21) Suppose $\mu : \kappa \to 2$ and $U \subseteq \wp(\kappa)$; define $\mu_U(A) := 1$ if $A \in U$ and $U_\mu := \{A\,;\, \mu(A) = 1\}$. Show that if U is a κ-complete nontrivial ultrafilter on κ, then μ_U is a κ-additive nontrivial measure on κ and if μ is a κ-additive nontrivial measure on κ, then U_μ is a κ-complete nontrivial ultrafilter on κ.

(22) Let κ be regular. Show that $\{X; |\kappa \backslash X| < \kappa\}$ is a κ-complete filter that is not an ultrafilter.

(23) Using the Axiom of Choice, show that every filter can be extended to an ultrafilter (preserving non-triviality).

(24) A model $(M, E) \models \mathsf{ZFC}$ is called an ω-model if its natural numbers are standard, i.e., if there is an isomorphism between $(\{x \in M\,;\, M \models \text{“}x \text{ is a natural number”}\}, E)$ and $(\omega, \in)$. Let M be an ω-model; without loss of generality, we can assume that $\omega \subseteq M$. We encode formulas of first-order logic by natural numbers, writing $\ulcorner\varphi\urcorner$ for the number coding φ. Let Φ be a set of first-order sentences such that Φ exists in M, i.e., there is some $x \in M$ such that $\varphi \in \Phi$ if and only if $M \models \ulcorner\varphi\urcorner \in x$. Show that Φ is consistent if and only if $M \models$ “Φ is consistent”. Deduce that if $\mathsf{ZFC} + \mathrm{Cons}(\mathsf{ZFC})$ is consistent, it cannot show the existence of an ω-model.

(25) Find an $\mathcal{L}_{\omega_1,\omega}$ formula that characterises the ω-models of ZFC.

(26) Give a concrete uncountable collection of $\mathcal{L}_{\omega_1,\omega}$ sentences that is countably satisfiable, but not satisfiable.

(27) If κ is a strongly compact cardinal, the Keisler-Tarski theorem makes a statement about κ-complete filters on arbitrary sets X. What does the proof show if κ is only assumed to be weakly compact? Why is that useless?

[*Hint.* If $\lambda < \kappa$, which filters on λ can be κ-complete?]

(28) In a reflection argument, we used Keisler's Theorem on the Extension Property to show that below each weakly compact cardinal is an inaccessible by reflecting the property “κ is inaccessible”. Clearly, it cannot be possible to reflect the property “κ is weakly compact”. Explain where the argument breaks down if you try to prove this.

(29) Let $\infty\mathsf{IC}$ be the statement “for all ordinals α, there is $\kappa > \alpha$ such that κ is inaccessible”. Show that if κ is weakly compact, then $\mathbf{V}_\kappa \models \infty\mathsf{IC}$.

(30) Presentation Example. Suppose that κ is a measurable cardinal and U is a κ-complete ultrafilter on κ, and $\pi : \mathbf{V}_\kappa \to \mathrm{Ult}(\mathbf{V}_\kappa, U)$ is the ultrapower embedding, i.e., $\pi(x) := [\mathrm{c}_x]_U$. By Łoś's Theorem, π is an elementary embedding. Show that $\{\pi(x)\,;\, x \in \mathbf{V}_\kappa\}$ is isomorphic to $\mathbf{V}_\kappa$ and transitive in $\mathrm{Ult}(\mathbf{V}_\kappa, U)$, i.e., if $z \in \pi(x)$, then there is $y \in \mathbf{V}_\kappa$ such that $z = \pi(y)$.

Conclude that the order type of the ordinals of $\mathrm{Ult}(\mathbf{V}_\kappa, U)$ is not equal to κ and that therefore $\mathrm{Ult}(\mathbf{V}_\kappa, U)$ is not isomorphic to $\mathbf{V}_\kappa$.