Large Cardinals
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Model Solutions for Example Sheet #1

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(1) If α is an ordinal, the *order topology* on α is the topology generated by the basic open sets $L_\beta := \{\gamma \in \alpha\,;\, \gamma < \beta\}$ and $R_\beta := \{\gamma \in \alpha\,;\, \gamma > \beta\}$. Check that every successor ordinal $\beta + 1 \in \alpha$ is an isolated point in this topology and determine the neighbourhoods of a limit ordinal $\lambda \in \alpha$.

Let α and β be ordinals with their respective order topologies. Show that an increasing function $f : \alpha \to \beta$ is continuous if and only if for all limit ordinals $\lambda \in \alpha$, we have that $f(\lambda) = \bigcup\{f(\gamma)\,;\, \gamma < \lambda\}$.

Solution. Recall from basic topology that a map $f : X \to Y$ is continuous if and only if f is continuous at all $x \in X$, i.e. for all $V \subseteq Y$ open with $f(x) \in V$, there is an open $U \subseteq X$ containing x with $f[U] \subseteq V$. Now if α is an ordinal with the order topology, then the isolated points of alpha are precisely the successor ordinals $\gamma < \alpha$ (and zero), since $\{\gamma\} = (\bigcup \gamma, \gamma + 1)$. On the other hand limit points of α are limit ordinals $< \alpha$. Therefore any function $f : \alpha \to \beta$ between ordinals is by definition continuous at successor ordinals and zero, and is continuous if and only if it is continuous at limit ordinals.

By the assumption on f, we know that $f(\lambda) \supseteq \bigcup\{f(\gamma)\,;\, \gamma < \lambda\}$. So, let $\lambda < \alpha$ be a limit and suppose that $f(\lambda) = \bigcup\{f(\gamma) : \gamma < \lambda\}$. If $V \subseteq \beta$ is open with $f(\lambda) \in V$, then w.l.g $V = (\mu, f(\lambda)]$ with $\mu < f(\lambda)$. Find $\gamma < \lambda$ with $\mu < f(\gamma) < f(\lambda)$. Then $(\gamma, \lambda]$ is open, and $f[(\gamma, \lambda]] \subseteq (\mu, f(\lambda)]$. Conversely, suppose that f is continuous. Let $x < f(\lambda)$, and consider the open set $(x, f(\lambda)]$. By continuity, there is some open $U \subseteq \alpha$ containing λ with $f[U] \subseteq (x, f(\lambda)]$. So, there is some $\gamma < \lambda$ such that $f[(\gamma, \lambda]] \subseteq (x, f(\lambda)]$. Therefore $x < f(\gamma + 1)$, and so $x \in \bigcup\{f(\gamma) : \gamma < \lambda\}$. ⊣

(2) In Lecture I, we stated that all normal ordinal operations have arbitrarily large fixed points. Prove that claim.

Solution. Let $F : \mathbf{Ord} \to \mathbf{Ord}$ be a normal ordinal operation, and let λ be an arbitrary ordinal. Define by recursion on ω: $A_0 := \lambda$, $A_{n+1} := F(A_n)$. Take $A = \bigcup\{A_n : n < \omega\}$ by Replacement and Union. Note that by definition each A_n is an ordinal, hence A is a union of a set of ordinals and therefore also an ordinal. We argue that $F(A) = A$. Indeed, if $A = \emptyset$ then $F(\emptyset) = \emptyset$. If A is a successor ordinal then $A = A_{n_0}$ for some $n_0 < \omega$, so $F(A) = F(A_{n_0}) = A_{n_0+1} = A$. If A is a limit ordinal then $f(A) = \bigcup\{f(a) : a < A\} = \bigcup\{f(A_n) : n < \omega\} = A$. So A is a fixed point of F. Furthermore $\lambda = A_0 \subseteq A$, so A can be chosen to be arbitrarily large. ⊣

(3) We call the axiom system that contains all axioms of ZFC except for the Axiom of Infinity FST (for "Finite Set Theory"). Consider the property $I(\alpha)$ defined by "α is a limit ordinal and $\alpha \neq 0$". Show that the property I is a *large cardinal property* for FST in the following sense:

If FST is consistent, then FST does not prove the existence of a cardinal with property I.

Solution. Suppose that $\mathsf{FST} \vdash \exists\alpha I(\alpha)$. Since $\mathsf{FST} + \exists\alpha I(\alpha) \vdash$ Infinity, we know by modus ponens that $\mathsf{FST} \vdash$ Infinity, and hence $\mathsf{FST} \vdash$ ZFC.

However $\mathsf{ZFC} \vdash \mathrm{Con}(\mathsf{FST})$. Indeed, we argue that $\mathbf{V}_\omega$ is a model of FST. All but the Replacement scheme can be easily verified. For this, observe that if $F : \mathbf{V}_\omega \to \mathbf{V}_\omega$ is a function and $x \in \mathbf{V}_\omega$, then for all $y \in x$, $\mathrm{rank}(F(y)) < \omega$. Then $C := \{\mathrm{rank}(F(y)) : y \in x\} \subseteq \omega$ is a finite set, so rank$(F[x]) \leq \sup C + 1 < \omega$. Therefore $F[x] \in \mathbf{V}_\omega$. Therefore $\mathsf{FST} \vdash \mathrm{Con}(\mathsf{FST})$, which contradicts Gödel's Second Incompleteness Theorem. $\dashv$

(4) Let λ and μ be limit ordinals and $f : \mu \to \lambda$ be a function. The function f is called *cofinal in* λ if $\mathrm{ran}(f)$ is a cofinal subset of λ. Show that

$$\begin{aligned}\mathrm{cf}(\lambda) &= \min\{\mu, ; \text{there is a cofinal function with domain } \mu\}\\ &= \min\{\mu\,; \text{there is a strictly increasing cofinal function with domain } \mu\}.\end{aligned}$$

Conclude that $\mathrm{cf}(\mathrm{cf}(\lambda)) = \mathrm{cf}(\lambda)$.

Solution. For the first equality, note that if $C \subseteq \lambda$ is cofinal, then the map $f : |C| \to \lambda$ given by the composition of a bijection between C and $|C|$ and inclusion is cofinal. Conversely, if there is $\alpha < \mathrm{cf}(\lambda)$ with $f : \alpha \to \lambda$ cofinal, then $f[a] \subseteq \lambda$ is cofinal and $|f[a]| \leq |a| < \mathrm{cf}(\lambda)$, contradiction.

The second equality follows from the fact that given $f : \mathrm{cf}(\lambda) \to \lambda$ cofinal, there is some $g : \mathrm{cf}(\lambda) \to \lambda$ strictly increasing and cofinal. Indeed, define $g : \mathrm{cf}(\lambda) \to \lambda$ by $\beta \mapsto \sup_{\delta<\beta}(f(\delta) + \beta)$. This is clearly strictly increasing and also maps into λ: if $\lambda = g(\beta)$ for some $\beta < \mathrm{cf}(\lambda)$ then $\lambda = \bigcup_{\delta<\beta}(f(\delta) + \beta)$, contradicting that $\beta < \mathrm{cf}(\lambda)$. Finally, it is easy to see that g is cofinal: if $\alpha < \lambda$ then $\exists \beta < \mathrm{cf}(\lambda)$ such that $a < f(\beta) \leq g(\beta + 1) < \lambda$.

Clearly, $\mathrm{cf}(\mathrm{cf}(\alpha)) \leq \mathrm{cf}(\alpha)$. For the other direction, pick $f : \mathrm{cf}(\mathrm{cf}(\alpha)) \to \mathrm{cf}(\alpha)$ and $g : \mathrm{cf}(\alpha) \to \alpha$ strictly increasing and cofinal. Their composition is a strictly increasing and cofinal map $\mathrm{cf}(\mathrm{cf}(\alpha)) \to \alpha$, and so $\mathrm{cf}(\alpha) \leq \mathrm{cf}(\mathrm{cf}(\alpha))$ by the above. $\dashv$

(5) Presentation Example. Let κ be regular, η be any ordinal and $f : \kappa \to \eta$ a strictly increasing function. Define $\lambda := \bigcup \mathrm{ran}(f)$. Show that $\mathrm{cf}(\lambda) = \kappa$. Conclude that $\mathrm{cf}(\aleph_\lambda) = \mathrm{cf}(\lambda)$.

Solution. Fix some cofinal map $g : \mathrm{cf}(\lambda) \to \lambda$. Consider the map $h : \mathrm{cf}(\lambda) \to \kappa$ given by mapping $\alpha < \mathrm{cf}(\lambda)$ to the least $\beta < \kappa$ such that $g(\alpha) < f(\beta)$. This is well-defined and cofinal. Indeed, if $\gamma < \kappa$, then find some $\alpha < \mathrm{cf}(\lambda)$ with $g(\alpha) > f(\gamma)$. Since f is strictly increasing, the least β with $f(\gamma) < g(\alpha) < f(\beta)$ must be strictly greater than γ, so $h(\alpha) > \gamma$. It follows that $\mathrm{cf}(\lambda) = \kappa$ by regularity of κ.

Finally, we show that $\mathrm{cf}(\aleph_\lambda) = \mathrm{cf}(\lambda)$. Take a strictly increasing cofinal map $\mathrm{cf}(\lambda) \to \lambda$ and compose it with $\alpha \mapsto \aleph_\alpha$. Then we have a strictly increasing cofinal map $f : \mathrm{cf}(\lambda) \to \aleph_\lambda$, and $\aleph_\lambda = \bigcup \mathrm{ran}(f)$. Since $\mathrm{cf}(\lambda)$ is regular, we use the same argument to deduce that $\mathrm{cf}(\aleph_\lambda) = \mathrm{cf}(\lambda)$.

$\dashv$

(6) We said that a cardinal κ satisfies *second order replacement* if for all $G : \mathbf{V}_\kappa \to \mathbf{V}_\kappa$ and $x \in \mathbf{V}_\kappa$, the set $G[x] := \{G(y)\,; y \in x\} \in \mathbf{V}_\kappa$. In Lecture II, we showed that if κ is inaccessible, it satisfies second order replacement. Show the converse. (This is known as Shepherdson's Theorem.)

Solution. We ought to show that κ is a strongly limit and regular. Suppose for a contradiction that there is $\alpha < \kappa$ such that $2^\alpha \geq \kappa$. Then, there is a surjection $\mathcal{P}(\alpha) \to \kappa$. Since $\alpha < \kappa$ and κ is a limit ordinal, $\mathcal{P}(\alpha) \in \mathbf{V}_\kappa$. Therefore $f[\mathcal{P}(\alpha)] = \kappa \in \mathbf{V}_\kappa$, which is a contradiction.

Furthermore, if $f : \alpha \to \kappa$ is cofinal with $\alpha < \kappa$ then since $\alpha \in \mathbf{V}_\kappa$, $f[\alpha] \in \mathbf{V}_\kappa$ by SOR. But then $\bigcup f[\alpha] \in \mathbf{V}_\kappa$, so there is some $\gamma < \kappa$ with $\sup_{\beta\in\alpha} f(\beta) < \gamma$, contradicting the cofinality of f. $\dashv$

(7) Let κ be a regular cardinal. If x is any set, we write $\mathrm{tcl}(x)$ for the transitive closure of x. Define $\mathbf{H}_\kappa := \{x\,; |\mathrm{tcl}(x)| < \kappa\}$. Show that $\mathbf{V}_\kappa = \mathbf{H}_\kappa$ if and only if κ is inaccessible.

Solution. We first show that $\mathbf{H}_\kappa \subseteq \mathbf{V}_\kappa$ for all infinite ordinals κ. We adapt the proof from Kenneth Kunen's Set Theory, p. 131. Let $x \in \mathbf{H}_\kappa$. We shall argue that $\mathrm{rank}(x) < \kappa$. Indeed, let $t = \mathrm{tcl}(x)$ and $S = \{\mathrm{rank}(y) : y \in t\} \subseteq \mathbf{Ord}$. Let α be the first ordinal not in S. By definition, this implies that $\alpha \subseteq S$. If $\alpha \neq S$, let β be the least element of S larger than α, and fix some $y \in x$ with $\mathrm{rank}(y) = \beta$. By transitivity of t, $\text{rank}(z) < \alpha$ for all $z \in y$, and so $\text{rank}(y) = \bigcup\{\text{rank}(z)+1 : z \in y\} \leq \alpha$, contradiction. So $S = \alpha$. Therefore $|t| < \kappa \implies \alpha < \kappa$, and so $\text{rank}(x) \leq \alpha < \kappa$.

Now, assuming that κ is inaccessible we have that $x \in \mathbf{V}_\kappa \implies x \in \mathbf{V}_\alpha$ for some $\alpha < \kappa$. By transitivity, $\text{tcl}(x) \subseteq \mathbf{V}_\alpha$ and therefore $|\text{tcl}(x)| \leq |\mathbf{V}_\alpha| = \alpha < \kappa$ since κ is inaccessible. Therefore $x \in \mathbf{H}_\kappa$. Conversely, suppose that $\mathbf{V}_\kappa = \mathbf{H}_\kappa$. Then if $\alpha < \kappa$, $\mathcal{P}(\alpha) \in \mathbf{V}_\kappa$ since κ is a limit ordinal, and therefore $2^\alpha \leq |\text{tcl}(\mathcal{P}(\alpha))| < \kappa$. So κ is a strong limit, and therefore inaccessible. $\dashv$

(8) Suppose that $(M, \in)$ and $(N, \in)$ are models of ZFC with $M \subseteq N$ and M is transitive in N. Show that the notions of "function", "injection", "surjection", "bijection", and "cofinal" are absolute between M and N.

Solution. We shall show that all of the above are expressible by Δ_0 formulae modulo ZFC. We leave out some of the details but they can be easily checked.

- $\text{fun}(f, a, b) := (f \subseteq a \times b) \wedge \forall(x \in a)\exists(z \in f)(x \in z) \wedge \forall(x \in a)\forall(z \in f)\forall(z' \in f)(x \in z \wedge x \in z' \to z = z')$
- $\text{inj}(f, a, b) := \text{fun}(f, a, b) \wedge \forall(x \in a)\forall(x' \in a)\forall(y \in b)(\{x, \{x, y\}\} \in f \wedge \{x', \{x', y\}\} \in f \to x = x')$
- $\text{sur}(f, a, b) := \text{fun}(f, a, b) \wedge \forall(y \in b)\exists(x \in a)(\{x, \{x, y\}\} \in f)$
- $\text{bij}(f, a, b) := \text{inj}(f, a, b) \wedge \text{sur}(f, a, b)$
- $\text{cof}(f, a, b) := \text{fun}(f, a, b) \wedge (a \in On) \wedge (b \in On) \wedge \forall(y \in b)\exists(x \in a)(y \in f(x))$

$\dashv$

(9) Let κ be inaccessible and $\lambda < \kappa$. Show that λ is inaccessible if and only if $\mathbf{V}_\kappa \models$ "λ is inaccessible".

Solution. Observe that "λ is a cardinal/regular/strong limit" are all Π_1 sentences modulo ZFC, i.e. ZFC proves that they are equivalent to formulas of the form $\forall\bar{x}\phi(\lambda, \bar{x})$ where all quantifiers in ϕ are bounded. All such universal properties are "downwards absolute", and so if they hold in $\mathbf{V}$ they must also hold in any $\mathbf{V}_\kappa$ that models ZFC. In particular, if $\lambda < \kappa$ and they are both inaccessible, then $\mathbf{V}_\kappa \models$ "λ is inaccessible".

Conversely, assume that $\mathbf{V}_\kappa \models$ "λ is inaccessible" and suppose for a contradiction that λ is not inaccessible. There are three possibilities with very similar arguments. If λ is not a cardinal, then there is some $\alpha < \lambda$ and a bijection $f : \alpha \to \lambda$. Since $\alpha < \lambda < \kappa$ we have that $f \subseteq \alpha \times \lambda \subseteq \mathbf{V}_{\lambda+2}$, and so $f \in \mathbf{V}_{\lambda+3} \subseteq \mathbf{V}_\kappa$ witnesses that λ is not a cardinal in $\mathbf{V}_\kappa$. If λ is not regular, then there is $\alpha < \lambda$ and a cofinal map $f : \alpha \to \lambda$. We argue similarly that $f \in \mathbf{V}_\kappa$ and witnesses that λ is not regular according to $\mathbf{V}_\kappa$. Finally, if f is not a strong limit, then there is $\alpha < \lambda$ with $2^\alpha \geq \lambda$. But κ is a strong limit and $2^\alpha < \kappa$, so $2^\alpha \in \mathbf{V}_\kappa$ showing that λ cannot be a strong limit in $\mathbf{V}_\kappa$. $\dashv$

(10) Show that every worldly cardinal is a limit cardinal.

[*Hint.* Use the fact that the proof of Hartogs's Lemma implies that there is a surjection from the power set of κ onto κ^+. If needed, refresh your memory of that proof.]

Solution. We saw in the lectures that κ is a cardinal, so it remains to show that it is a limit. Let $\alpha < \kappa$ be a cardinal. Then $\mathbf{V}_\kappa \models$ "$\exists$ ordinal $\beta > \alpha$ such that there is no surjection $\alpha \to \beta$". Let $\beta \in \mathbf{V}_\kappa$ witness this. Suppose for a contradiction that $f : \alpha \to \beta$ is a surjection. Then $f \subseteq \alpha \times \beta \subseteq \mathbf{V}_{\beta+2}$, so $f \in \mathbf{V}_{\beta+3} \subseteq \mathbf{V}_\kappa$ as κ is a limit ordinal. Since being a surjection is absolute, this is a contradiction. Hence there is really no surjection between α and β, and so $\alpha^+ \leq \beta \in \mathbf{V}_\kappa$. It follows that $\alpha^+ < \kappa$. $\dashv$

(11) Prove the *Tarski-Vaught Test* for being an elementary substructure as cited in Lecture III.

Solution. We ought to show that for L-structures $M \subseteq N$, $M \preccurlyeq N$ if and only if for all formulas $\phi(x, \bar{y})$ and $\bar{m} \in M$, $N \models \exists x\phi(x, \bar{m})$ implies that there is $a \in M$ such that $N \models \phi(a, \bar{m})$. Clearly this holds if the embedding is elementary.

Conversely, we show by induction on the structure of ϕ that for all formulas $\phi(\bar{y})$ and for all $\bar{m} \in M$,

$$M \models \phi(\bar{m}) \iff N \models \phi(\bar{m}).$$

We know that atomic formulas are preserved, while it is also easy to check $\phi = \neg\psi$ and $\phi = \psi \wedge \chi$. Suppose that $\phi(\bar{y}) = \exists x \psi(x, \bar{y})$. By induction, we may assume that that $\psi(x, \bar{y})$ is preserved. So, if $M \models \phi(\bar{m})$ then there is $a \in M$ such that $M \models \psi(a, \bar{m})$ and so by induction $N \models \psi(a, \bar{m})$. If on the other hand $N \models \phi(\bar{m})$ then by the assumption there is some $a \in M$ such that $N \models \psi(a, \bar{m})$ and by induction ψ is preserved, i.e. $M \models \psi(a, \bar{m})$. Therefore the claim follows, and so $M \preccurlyeq N$. ⊣

(12) Prove *Tarski's Chain Lemma* as formulated in Lecture IV.

Solution. Let $(M_l)_{l \in L}$ be an elementary chain indexed by some total order $(L, <)$. Consider $M := \bigcup_{l \in L} M_l$. We shall prove that $\forall l \in L, M_l \preccurlyeq M$, i.e. for all $l \in L$ and $\bar{m} \in M_l, M_l \models \phi(\bar{m}) \iff M \models \phi(\bar{m})$ by induction on the structure of ϕ. The proof is essentially exactly the same as with the Tarski-Vaught test, only now the induction is for all $l \in L$. What changes is the backwards direction of the existential case. Indeed, if $\phi(\bar{y}) = \exists x \psi(x, \bar{y})$ and $M \models \phi(\bar{m})$ for $\bar{m} \in M_l$, then there is some some $a \in M$ such that $M \models \psi(a, \bar{m})$. By definition, we can find some $l_a > l \in L$ such that $a \in M_{l_a}$. By induction $M_{l_a} \models \psi(a, \bar{m})$, so in particular $M_{l_a} \models \phi(\bar{m})$. But by assumption $M_l \preccurlyeq M_{l_a}$ so $M_l \models \phi(\bar{m})$, which concludes the proof. ⊣

(13) Let β be any ordinal and $R \subseteq \mathbf{V}_\beta$. An ordinal $\alpha < \beta$ is called an *R-Lévy ordinal for* β if $(\mathbf{V}_\alpha, \in, R \cap \mathbf{V}_\alpha)$ is an elementary substructure of $(\mathbf{V}_\beta, \in, R)$. Show that no α can be an R-Lévy ordinal for all $R \subseteq \mathbf{V}_\beta$.

Solution. Take $R = \mathbf{V}_\alpha$, and suppose that $(\mathbf{V}_\alpha, \in, \mathbf{V}_\alpha) \preccurlyeq (\mathbf{V}_\beta, \in, \mathbf{V}_\alpha)$. Since $\mathbf{V}_\beta \models \exists x(\neg R(x))$, then so does $\mathbf{V}_\alpha$. But this is a contradiction. ⊣

(14) Presentation Example. Show the following theorem due to Lévy: an ordinal κ is an inaccessible cardinal if and only if for each $R \subseteq \mathbf{V}_\kappa$ there is an R-Lévy ordinal for κ.

Solution. Suppose that κ is inaccessible, and let $R \subseteq \mathbf{V}_\kappa$. Define by recursion on ω: $\alpha_0 = \emptyset, \alpha_{n+1} =$ the least $\beta \geq \alpha_n$ such that whenever $y_1, \dots, y_k \in \mathbf{V}_{\alpha_n}$ and $(\mathbf{V}_\kappa, \in, R) \models \exists x \phi(x, y_1, \dots, y_k)$ for some formula ϕ, there is an $x_0 \in \mathbf{V}_\beta$ such that $(\mathbf{V}_\kappa, \in, R) \models \phi(x_0, y_1, \dots, y_k)$. Since κ is inaccessible, $|V_{\alpha_n}| < \kappa$ and so $\alpha_{n+1} < \kappa$. Finally take $\alpha = \bigcup_\omega \alpha_n$. Using Tarski-Vaught, we may easily verify that $(\mathbf{V}_\alpha, \in, R \cap \mathbf{V}_\alpha) \preccurlyeq (\mathbf{V}_\kappa, \in, R)$. Note that by starting with any arbitrary $\alpha_0 = \lambda < \kappa$, the above argument shows that $\{\alpha : (\mathbf{V}_\alpha, \in, R \cap \mathbf{V}_\alpha) \preccurlyeq (\mathbf{V}_\kappa, \in, R)\}$ is in fact unbounded in κ.

For the converse, notice first that κ must necessarily be infinite. If κ is not regular, then there is $\beta < \kappa$ and $f : \beta \to \kappa$ cofinal. Let $R = \{\beta\} \cup f$ and find $\alpha < \kappa$ such that $(\mathbf{V}_\alpha, \in, R \cap \mathbf{V}_\alpha) \preccurlyeq (\mathbf{V}_\kappa, \in, R)$. Since β is the only ordinal in R, we see that $\beta \in \mathbf{V}_\alpha$ by elementarity. But then there is some $\gamma < \beta$ in $\mathbf{V}_\alpha$ with $\alpha < f(\gamma) < \kappa$ and $f(\gamma) \in \mathbf{V}_\alpha$, contradiction.

Also, if κ is not a strong limit then we can find $\beta < \kappa$ with $2^\beta \geq \kappa$. Find a surjection $g : \mathcal{P}(\beta) \to \kappa$ and take $R = \{\beta + 1\} \cup g$. By assumption, there is $\alpha < \kappa$ such that $(\mathbf{V}_\alpha, \in, R \cap \mathbf{V}_\alpha) \preccurlyeq (\mathbf{V}_\kappa, \in, R)$. Since $\beta + 1 \in V_\alpha$, it follows that $\mathcal{P}(\beta) \in \mathbf{V}_\alpha$ and so again we can find some $x \in \mathcal{P}(\beta)$ such that $g(x) = \alpha \in \mathbf{V}_\alpha$, contradiction. ⊣