Large Cardinals
Lent Term 2022
Part III of the Mathematical Tripos
University of Cambridge
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Example Sheet #1

Examples Classes.
#1: Friday 11 February 2022, 3:30–5pm, **MR4**.
#2: Friday 25 February 2022, 3:30–5pm, **MR4**.
#3: Friday 18 March 2022, 3:30–5pm, **MR4**.

Presentation. Two of the examples are designed to be a Presentation Example (marked on the sheet). We encourage all students to meet in pairs, work together on these examples, and prepare a short presentation of their solutions that can be given on the blackboard in **MR4** during the examples class. The discussion during your meeting should be both about the mathematical content and about the preparation of the presentation.

Marking. You can submit all of your work to Ioannis Eleftheriadis (`ie257`) as a *single pdf file* by e-mail or hand it to him on paper during the examples class. Please submit all work before the start of the examples class. Work that is submitted at least 24 hours before the examples class could already be marked and returned during the examples class. We cannot guarantee that all work will be marked, but we shall endeavour to mark at least two examples per submission. Model solutions will be provided on the moodle page of the course.

(1) If α is an ordinal, the *order topology* on α is the topology generated by the basic open sets $L_\beta := \{\gamma \in \alpha\,;\, \gamma < \beta\}$ and $R_\beta := \{\gamma \in \alpha\,;\, \gamma > \beta\}$. Check that every successor ordinal $\beta + 1 \in \alpha$ is an isolated point in this topology and determine the neighbourhoods of a limit ordinal $\lambda \in \alpha$.

Let α and β be ordinals with their respective order topologies. Show that an increasing function $f : \alpha \to \beta$ is continuous if and only if for all limit ordinals $\lambda \in \alpha$, we have that $f(\lambda) = \bigcup\{f(\gamma)\,;\, \gamma < \lambda\}$.

(2) In Lecture I, we stated that all normal ordinal operations have arbitrarily large fixed points. Prove that claim.

(3) We call the axiom system that contains all axioms of ZFC except for the Axiom of Infinity FST (for "Finite Set Theory"). Consider the property $I(\alpha)$ defined by "α is a limit ordinal and $\alpha \neq 0$". Show that the property I is a *large cardinal property* for FST in the following sense:

If FST is consistent, then FST does not prove the existence of a cardinal with property I.

(4) Let λ and μ be limit ordinals and $f : \mu \to \lambda$ be a function. The function f is called *cofinal in* λ if $\mathrm{ran}(f)$ is a cofinal subset of λ. Show that

$$\begin{aligned}\mathrm{cf}(\lambda) &= \min\{\mu, ; \text{there is a cofinal function with domain } \mu\} \\ &= \min\{\mu\,; \text{there is a strictly increasing cofinal function with domain } \mu\}.\end{aligned}$$

Conclude that $\mathrm{cf}(\mathrm{cf}(\lambda)) = \mathrm{cf}(\lambda)$.

(5) Presentation Example. Let κ be regular, η be any ordinal and $f : \kappa \to \eta$ a strictly increasing function. Define $\lambda := \bigcup \mathrm{ran}(f)$. Show that $\mathrm{cf}(\lambda) = \kappa$. Conclude that $\mathrm{cf}(\aleph_\lambda) = \mathrm{cf}(\lambda)$.

(6) We said that a cardinal κ satisfies *second order replacement* if for all $G : \mathbf{V}_\kappa \to \mathbf{V}_\kappa$ and $x \in \mathbf{V}_\kappa$, the set $G[x] := \{G(y)\,; y \in x\} \in \mathbf{V}_\kappa$. In Lecture II, we showed that if κ is inaccessible, it satisfies second order replacement. Show the converse. (This is known as Shepherdson's Theorem.)

(7) Let κ be a regular cardinal. If x is any set, we write $\mathrm{tcl}(x)$ for the transitive closure of x. Define $\mathbf{H}_\kappa := \{x\,; |\mathrm{tcl}(x)| < \kappa\}$. Show that $\mathbf{V}_\kappa = \mathbf{H}_\kappa$ if and only if κ is inaccessible.

(8) Suppose that $(M, \in)$ and $(N, \in)$ are models of ZFC with $M \subseteq N$ and M is transitive in N. Show that the notions of "function", "injection", "surjection", "bijection", and "cofinal" are absolute between M and N.

(9) Let κ be inaccessible and $\lambda < \kappa$. Show that λ is inaccessible if and only if $\mathbf{V}_\kappa \models$ "λ is inaccessible".

(10) Show that every worldly cardinal is a limit cardinal.

[*Hint.* Use the fact that the proof of Hartogs's Lemma implies that there is a surjection from the power set of κ onto κ^+. If needed, refresh your memory of that proof.]

(11) Prove the *Tarski-Vaught Test* for being an elementary substructure as cited in Lecture III.

(12) Prove *Tarski's Chain Lemma* as formulated in Lecture IV.

(13) Let β be any ordinal and $R \subseteq \mathbf{V}_\beta$. An ordinal $\alpha < \beta$ is called an *R-Lévy ordinal for* β if $(\mathbf{V}_\alpha, \in, R \cap \mathbf{V}_\alpha)$ is an elementary substructure of $(\mathbf{V}_\beta, \in, R)$. Show that no α can be an R-Lévy ordinal for all $R \subseteq \mathbf{V}_\beta$.

(14) Presentation Example. Show the following theorem due to Lévy: an ordinal κ is an inaccessible cardinal if and only if for each $R \subseteq \mathbf{V}_\kappa$ there is an R-Lévy ordinal for κ.