Lectio Ultima

XXIV

Twenty-forth & final lecture

LENT 2021 17 MARCH 2021

INFINITE GAMES

Theorem (Solovay).

$AD \implies \aleph_1$ is measurable.

Already done:

① $AD \implies$ every ultrafilter is $\aleph_1$-complete

② The cone filter F_D on ω^ω defined by $A \in F_D \iff \exists x \; \mathrm{Cone}(x) \subseteq A$ is a filter.

$\mathcal{D}_D$ definability degrees

$\omega^\omega / \equiv_D$

if $Z \subseteq \mathcal{D}_D$, then $\bigcup Z \subseteq \omega^\omega$.

The set $\bigcup Z$ is $\equiv_D$-invariant,

i.e., if $x \in \bigcup Z$ and $y \equiv_D x$, then $y \in \bigcup Z$.

Being $\equiv_D$-invariant just means being a union of $\equiv_D$-equivalence classes.

The operation

$$Z \longmapsto \bigcup Z$$
$$\wp(\mathcal{D}_D) \longrightarrow \wp(\omega^\omega)$$

can be reversed (on the $\equiv_D$-invariant sets) by

$$A \longmapsto \{d \in \mathcal{D}_D \,;\, d \subseteq A\}.$$

<u>Thus</u>: subsets of $\mathcal{D}_D$ are $\equiv_D$-invariant sets in disguise.

<u>RELATIONSHIP</u> <u>BETWEEN</u> <u>STRATEGIES</u> <u>& $\mathcal{D}_D$</u>:

Take your favourite definable bijection $\pi : \underline{\omega} \longrightarrow \omega^{<\omega}$.
Then we can code strategies:

if σ is a strategy

$$\mathrm{code}(\sigma) : \begin{array}{ccc} n & \longmapsto & \sigma(\pi(n)) \\ \omega & \longrightarrow & \omega \end{array}$$

then $\mathrm{code}(\sigma) \in \omega^\omega$ and we have that

$$\sigma \equiv_D \mathrm{code}(\sigma).$$

If we play σ against τ

$$\sigma * \tau \leq_D \underbrace{\mathrm{code}(\sigma) * \mathrm{code}(\tau)}_{\text{LEAST UPPER BOUND OF}} \quad \begin{array}{c}\mathrm{code}(\sigma) \\ \& \\ \mathrm{code}(\tau)\end{array}$$

Let's do $\sigma * x$:

This is the unique element $z \in \omega^\omega$ s.t. for all k

$$z(2k+1) = x(k)$$

and

$$z(2k) = \mathrm{code}(\sigma)(\pi^{-1}(z \upharpoonright 2k))$$

This formula witnesses that

$$\sigma * x \leq_D \mathrm{code}(\sigma) * x .$$

[In general, $\sigma * x \not\equiv_D \mathrm{code}(\sigma) * x$.]

<u>Martin's Lemma</u> Suppose A is $\equiv_D$-invariant Then

(i) if player I wins $G(A)$, then $A \in F_D$

(ii) if player II wins $G(A)$, then $\omega^\omega \setminus A \in F_D$.

<u>Corollary</u>. AD $\Longrightarrow$ the filter on $\mathcal{D}_D$ defined by $Z \in U_M :\Longleftrightarrow \bigcup Z \in F_D$

<u>MARTIN MEASURE</u>

is an ultrafilter.

Proof of Martin's Lemma. We're just going to see the case of player I. Please check for yourself that the case for player II is the same argument.

Suppose $A \subseteq \omega^\omega$ is $\equiv_D$-invariant and σ is winning for player I in $G(A)$.

Let $c := \mathrm{code}(\sigma)$ and let's claim:

$$\mathrm{Cone}(c) \subseteq A.$$

So take any $\underbrace{x \in \mathrm{Cone}(c) = \{y \in \omega^\omega ;\ c \le_D y\}}_{c \le_D x}$

Whenever $a \le_D b$, then $a * b \equiv_D b$ [since we proved that $a*b$ is the least upper bound].

So here: $c \le_D x \equiv_D c * x = \mathrm{code}(\sigma) * x$.

We had

$$\begin{array}{ccccc} \sigma * x & \le_D & \mathrm{code}(\sigma) * x & = & c * x \\ \ge_D & & \equiv_D & & \\ x & & x & & \end{array}$$

$$\begin{array}{ccc} \Longrightarrow \sigma * x \equiv_D x. & \Longrightarrow & x \in A \\ A \ni & & \text{since } A \text{ was } \equiv_D\text{-invariant} \end{array}$$

q.e.d.

Proof of Solovay's Theorem.

We have established that U_M is an ultrafilter on $\mathcal{D}_D$.

Idea Give a function $f : \mathcal{D}_D \longrightarrow \aleph_1$ s.t. $f_* U_M$ is a non-principal ultrafilter on $\aleph_1$.

Clear No matter f is, $f_* U_M$ is going to be an $\aleph_1$-complete ultrafilter.

So The crucial part is to find f s.t. $f_* U_M$ is non-principal.

Important: If $d \in \mathcal{D}_D$, then d is a countable set of elements of ω^ω. $d = [x]_{\equiv_D}$

$$W_x := \{\alpha \,;\, \exists y \,(y \leq_D x \wedge y \in WF_\alpha)\}$$

This is a countable set. Clearly if $x \equiv_D x'$, then $W_x = W_{x'}$.

$$W_d := W_x \text{ for any } x \in d.$$

By $AC_\omega(\mathbb{R})$, $\aleph_1$ is regular, thus

$$\alpha_d := \sup W_d < \omega_1.$$

$\alpha_x := \sup W_x$

[For all $x \in d$, $\alpha_x = \alpha_d$.]

Define

$$f : \mathcal{D}_D \longrightarrow \aleph_1$$
$$d \longmapsto \alpha_d .$$

Then $f_* U_M$ is an $\aleph_1$-complete uf. on $\aleph_1$.

<u>CLAIM</u> $f_* U_M$ is not principal.

Suppose it is: $\{\gamma\} \in f_* U_M$

$$\iff \{d \in \mathcal{D}_D \,;\, \alpha_d = \gamma\} \in U_M$$

$$\iff \{x \in \omega^\omega \,;\, \alpha_x = \gamma\} \in F_D$$

$\iff$ there is z s.t.

$$\mathrm{Cone}(z) \subseteq \{x \in \omega^\omega \,;\, \alpha_x = \gamma\}$$

Consider some $y \in WF_{\gamma+1}$. Clearly,

$$y \notin \{x \in \omega^\omega \,;\, \alpha_x = \gamma\}.$$

Now $\underline{z * y \in \mathrm{Cone}(z)}$, but $y \leq_D z * y$,

so $\gamma + 1 \in W_{z*y}$, so $\underline{\alpha_{z*y} \neq \gamma}$.

This is a contradiction! q.e.d.

CH : $2^{\aleph_0} = \aleph_1$

($\Rightarrow$ $\mathbb{R}$ are wellordered)

CH* : every $A \subseteq \mathbb{R}$ is either ctble or in bij. with the reals

AD $\rightarrow$ PSP $\Longrightarrow$ CH*

AD $\Rightarrow$ there is a surjection from $2^{\aleph_0}$ onto $2^{\aleph_1}$. $\leftarrow$ ES#4 (50)

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no inj. from $\aleph_1 \longrightarrow \omega^\omega$ is ES#4 (43)

Solovay's original proof (49).

NOTES FROM THE DISCUSSION AFTER THE LECTURE