LECTIO PENULTIMA

XXIII

INFINITE GAMES
LENT 2021 / 15 MARCH 2021

Theorem (Solovay)

$$AD \Longrightarrow \aleph_1 \text{ is measurable.}$$

ALREADY: $AD \Longrightarrow$ every ultrafilter is $\aleph_1$-complete

Still needed: There is a non-principal uf on $\aleph_1$.

Basic idea: Have ultrafilter U on X and fn $f: X \longrightarrow \aleph_1$ and consider $f_* U$ which is uf. on $\aleph_1$.

Properties get transferred except for non-principality.

First idea: Consider $f : WF \longrightarrow \aleph_1$
$x \longmapsto \|x\|$.

If we could use AD to define a game on WF s.t. it defines an ultrafilter U on WF and $f_* U$ is non-principal, we'd be done.

This is Solovay's original idea and will be on Example Sheet #4.

We're going to see a slightly different proof, due to Tony Martin.

Consider $V_{\omega+1}$, the $\omega+1$-st level of the von Neumann hierarchy: $\mathcal{P}(V_\omega)$.

Elements of $V_{\omega+1}$ are:

(1) if $x \in \omega^\omega$, then $x \in V_{\omega+1}$

Elements of x are $(n,m) \in \omega \times \omega$

$$(n,m) = \{\{n\},\{n,m\}\}$$

rank $\leq \max(n+2, m+2) \in \mathbb{N}$

So if $p \in x$, then $p \in V_\omega$,

so $x \in \mathcal{P}(V_\omega) = V_{\omega+1}$.

(2) Similarly $\omega^{<\omega}$, but also STRATEGIES

A strategy $\sigma : \omega^{<\omega} \longrightarrow \omega$

$p \in \sigma \rightsquigarrow p = (s, n)$

$s \in V_\omega$, $n \in V_\omega$

$\xrightarrow[\text{same argument}]{}$ $p \in V_\omega$

$\Longrightarrow \sigma \in \mathcal{P}(V_\omega) = V_{\omega+1}$

<u>Definition</u> Let $x, y \in V_{\omega+1}$.

<u>Define</u> $x \leq_D y$ $:\Longleftrightarrow$ there is a formula φ s.t.

$$(V_{\omega+1}, \in) \models \varphi(w, y) \Longleftrightarrow \underline{w \in x}$$

<u>x is definable from y</u>

<u>Properties</u> (1) $\leq_D$ is reflexive:

$$x \leq_D x \quad [\varphi(w, z) := w \in z]$$

(2) $\leq_D$ is transitive:

$$x \leq_D y \text{ and } y \leq_D z \Rightarrow x \leq_D z$$

[Let φ, ψ be formulas witnessing $x \leq_D y$ and $y \leq_D z$, respectively.

$$w \in x \Longleftrightarrow V_{\omega+1} \models \varphi(w, y)$$

$$(*) \quad w \in y \Longleftrightarrow V_{\omega+1} \models \psi(w, z)$$

Formula: $\exists u \; \forall v (v \in u \longleftrightarrow \psi(v, z)) \wedge \varphi(w, u)$

This formula defines x from z.]

(3) If $x \in V_{\omega+1}$, then

$$\{y \in V_{\omega+1} ; y \leq_D x\}$$

is countable since there are only countably many formulas witnessing $\leq_D$.

(4) $\leq_D$ has no largest element.

(5) If x is definable without parameters, i.e. $w \in x \iff V_{\omega+1} \models \varphi(w)$.

then for any y, $x \leq_D y$.

Thus: these objects are all minimal in $\leq_D$.

(6) $\leq_D$ is not antisymmetric:

If x, y are both definable w/o parameters, but $x \neq y$, then by (5) $x \leq_D y$ and $y \leq_D x$, but $x \neq y$.

E.g., $\emptyset, \{\emptyset\} \in V_{\omega+1}$ $\quad \emptyset \neq \{\emptyset\}$.

$$w \in \emptyset \iff V_{\omega+1} \models w \neq w.$$

$$w \in \{\emptyset\} \iff V_{\omega+1} \models \forall v\, (v \notin w).$$

Def. A relation $\leq$ on X is called a **preorder** if $\leq$ is reflexive and transitive.

If $\leq$ is a preorder, then

$$x \equiv y \;:\Longleftrightarrow\; x \leq y \wedge y \leq x$$

is an equivalence relation. Consider $Q := X/_{\equiv}$ and define $\leq$ on Q by

$$[x]_{\equiv} \leq [y]_{\equiv} \;:\Longleftrightarrow\; x \leq y.$$

Need to check that this is well-defined:

$$\underline{x \equiv x', y \equiv y', x \leq y}$$

$$x' \leq x \leq y \leq y' \implies x' \leq y'.$$

Applied to $\leq_D$:

$$\left(V_{\omega+1}/_{\equiv_D}, \leq_D\right) \text{ is a partial order}$$

We saw $\omega^\omega \subseteq V_{\omega+1}$.

$$\underbrace{\left(\omega^\omega/\equiv_D, \leq_D\right)}_{\mathcal{D}_D}$$ is called the partial order of <u>DEFINABILITY DEGREES</u>

$\left[\; x \leq_D y \iff \text{there is } \varphi \text{ s.t. } w \in x \iff V_{\omega+1} \models \varphi(w, y)\right.$

if $x, y \in \omega^\omega$, this makes sense. $\left.\right]$

$x, y \in \omega^\omega$

$$x * y(k) := \begin{cases} x(n) & 2n = k \\ y(n) & 2n+1 = k \end{cases}$$

If σ_x, σ_y are the blindfolded strategies for x, y then $\sigma_x * \sigma_y = x * y$.

$x \longmapsto x_{\mathrm{I}}$

$x \longmapsto x_{\mathrm{II}}$

For every x,

$$x_{\mathrm{I}} \leq_D x$$

$$x_{\mathrm{II}} \leq_D x$$

Thus $x \leq_D x * y$ and $y \leq_D x * y$

We'll see: $x * y$ is the least upper bound of x and y in $\leq_D$.

Prop. If $x, y \in \omega^\omega$ and z is s.t.

(1) $x \leq_D z$ and

(2) $y \leq_D z$

then $x * y \leq_D z$.

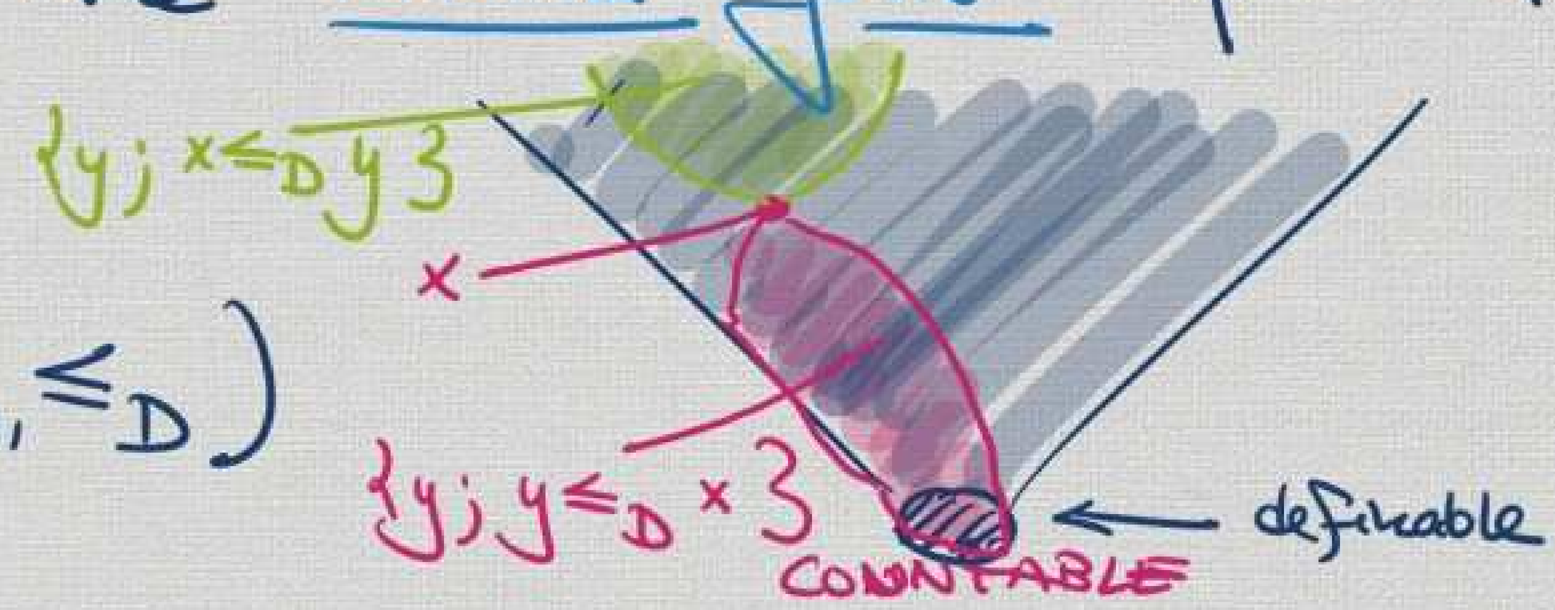
Proof. (1) $w \in x \iff V_{\omega+1} \models \varphi(w, z)$

(2) $w \in y \iff V_{\omega+1} \models \psi(w, z)$

$$\underset{(k,l)}{w} \in x * y \iff \left[\exists k, l \ \varphi((k,l), z) \wedge w = (2k, l)\right] \vee \left[\exists k, l \ \psi((k,l), z) \wedge w = (2k+1, l)\right].$$

q.e.d.

Operation $x, y \longmapsto x * y$ is also known as the TURING JOIN of x and y.



$(\mathcal{D}_D, \leq_D)$

Consider now

$$\mathrm{Cone}(x) := \{ y \in \omega^\omega ; \; x \leq_D y \}$$

the cone of x

If $z \neq z' \in \omega^\omega$, we get that

$$x \leq_D x * z \quad \text{and} \quad x \leq_D x * z'$$

and $x * z \neq x * z'$,

so $|\mathrm{Cone}(x)| = 2^{\aleph_0}$.

Definition We define the cone filter F_D on ω^ω by

$$A \in F_D \; :\Longleftrightarrow \text{ there is an } x \in \omega^\omega \text{ s.t. } \mathrm{Cone}(x) \subseteq A.$$

Let's check the filter properties:

Clearly $\emptyset \notin F_D$. Trivial.

$\omega^\omega \in F_D$.

If $A \in F_D$ and $B \supseteq A$, then $B \in F_D$.

Let $A, B \in F_D$. This means:

there are x, y s.t.

$$\mathrm{Cone}(x) \subseteq A$$
$$\mathrm{Cone}(y) \subseteq B.$$

Define $z := x * y$.

<u>CLAIM</u> $\mathrm{Cone}(z) \subseteq A \cap B$.

If $z \leq_D w \implies x \leq_D z \leq_D w$

$\implies w \in \mathrm{Cone}(x)$

$\implies w \in A$

$\Downarrow$

$y \leq_D z \leq_D w$

$\implies w \in \mathrm{Cone}(y)$

$\implies w \in B \implies w \in A \cap B$.

<u>SUMMARY</u> F_D is a filter on ω^ω.