Twenty-second Lecture

12 March 2021

XXII

INFINITE GAMES
LENT 2021

Goal: $\boxed{\text{Theorem (Solovay). } AD \Longrightarrow \aleph_1 \text{ is measurable}}$

Theorem (ZFC) Every filter can be extended to an ultrafilter.

Zorn's Lemma

- If F that contains the complements of the singletons, then $U \supseteq F$ ultra cannot be principal.
- So if $F := \{A \subseteq X \,;\, X \setminus A \text{ is finite}\}$ is the Fréchet filter (cofinite filter) and $U \supseteq F$, then U is a nonprincipal uf.

Suppose $\pi : X \longrightarrow Y$ any map and F is a filter on X, we can define $\pi_* F := \{A \subseteq Y \,;\, f^{-1}[A] \in F\}$, the image filter.

It is a filter:
- $f^{-1}[\emptyset] = \emptyset \notin F$
- $f^{-1}[Y] = X \in F$
- $f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B]$
- $A \subseteq B \Longrightarrow f^{-1}[A] \subseteq f^{-1}[B]$

$*$

Furthermore:

- if F was ultra, then $\pi_* F$ is ultra

  $$\left[\ \pi^{-1}[X \setminus A] = Y \setminus \pi^{-1}[A]\ \right]$$

- if F was λ-complete, then so is $\pi_* F$

  $$\left[\ \pi^{-1}\Big[\bigcap_{\alpha<\gamma} X_\alpha\Big] = \bigcap_{\alpha<\gamma} \pi^{-1}[X_\alpha]\ \right].$$

In general, non-principality is <u>not</u> preserved by images:

if π is a constant function

$$\pi(x) = a \in Y \quad (\text{f.a. } x \in X).$$

$$\pi_* F = \{ A \subseteq Y \,;\, \underbrace{\pi^{-1}[A]}_{\substack{\text{either } X \ [\text{if } a \in A] \\ \text{or } \emptyset \ [\text{if } a \notin A]}} \in F \}$$

$$= \{ A \subseteq Y \,;\, a \in A \}.$$

<u>Remark</u>: The main obstacle later is to show non-principality of image ultrafilters.

<u>Lemma</u> (ZF)

If there is an ultrafilter U on X that is not $\aleph_1$-complete, then there is a non-principal uf. on $\mathbb{N}$.

<u>Proof.</u> By assumption U is not $\aleph_1$-complete, so we can write

$$X = \bigcup_{n \in \mathbb{N}} X_n$$

s.t. the X_n are pairwise disjoint and $X_n \notin U$ f.e. $n \in \mathbb{N}$.

$$\pi : X \longrightarrow \mathbb{N}$$
$$x \longmapsto n \text{ if } x \in X_n .$$

By our general theory of image filters, we get that $\pi_* U$ is an ultrafilter on $\mathbb{N}$.

<u>Claim</u> $\pi_* U$ is non-principal.

Suppose not: $\{k\} \in \pi_* U$

$$\iff \underbrace{\pi^{-1}[\{k\}]}_{= X_k} \in U$$

This is a contradiction. q.e.d.

Corollary ZF + there is no non-principal uf. on $\mathbb{N}$ $\Longrightarrow$ every ultrafilter is $\aleph_1$-complete.

Theorem AD $\Longrightarrow$ there is no non-principal ultrafilter on $\mathbb{N}$.

Corollary AD $\Longrightarrow$ every ultrafilter is $\aleph_1$-complete.

Proof of Thm. Technique of strategy stealing: assume I/II has a w.s. and use it to define a w.s. for II/I.

Assume U is a non-principal uf. on $\mathbb{N}$ and define a non-determined game.

Moves are $s \in [\mathbb{N}]^{<\omega}$

G_U [use definable bijection between $\mathbb{N}$ and $[\mathbb{N}]^{<\omega}$ to see that AD implies the determinacy of games like this]

$$\begin{array}{lcccccccl} \mathrm{I} & s_0 & & s_2 & & s_4 & & \cdots \longrightarrow & A_{\mathrm{I}} := \bigcup_{i \in \mathbb{N}} s_{2i} \\ \mathrm{II} & & s_1 & & s_3 & & s_5 & \cdots \longrightarrow & A_{\mathrm{II}} := \bigcup_{i \in \mathbb{N}} s_{2i+1} \end{array}$$

Rule: s_n has to be disjoint from $\bigcup_{i<n} s_i$.

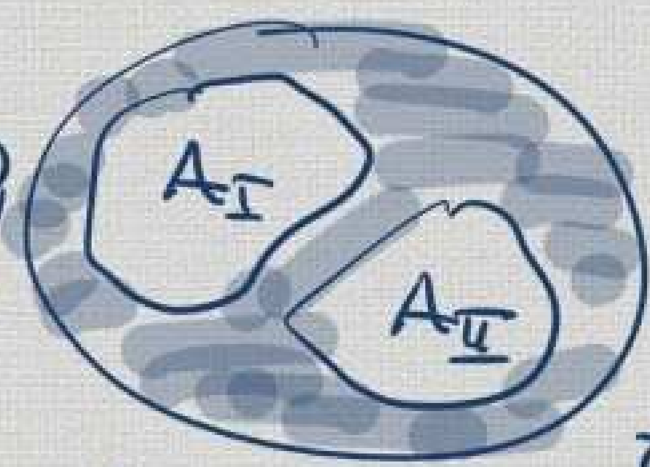
If you are the first one to break the rule, you lose. If no one broke the rule: $A_{\mathrm{I}} \cap A_{\mathrm{II}} = \emptyset$

$$A_{\mathrm{I}} := \bigcup_{i\in\mathbb{N}} s_{2i} \qquad A_{\mathrm{II}} := \bigcup_{i\in\mathbb{N}} s_{2i+1}$$

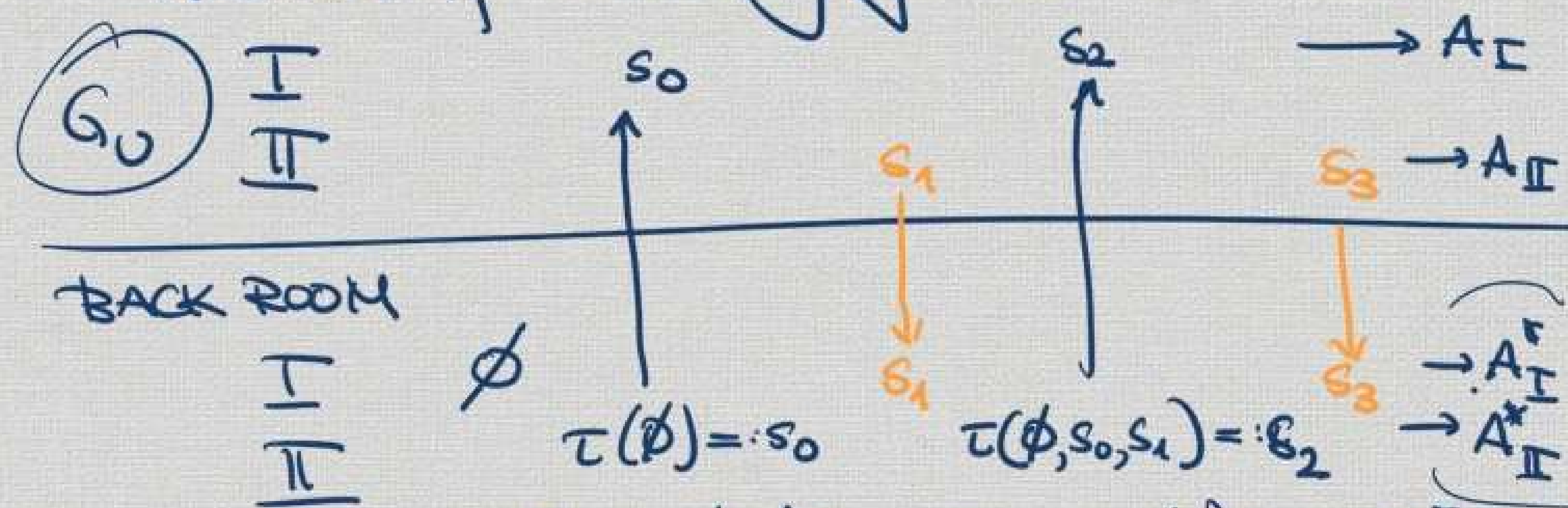
$$A_{\mathrm{I}} \cap A_{\mathrm{II}} = \emptyset$$

Player I wins if $A_{\mathrm{I}} \in \mathcal{U}$.

[Note that if II wins, it means that $\mathbb{N}\setminus A_{\mathrm{I}} \in \mathcal{U}$, but not necessarily that $A_{\mathrm{II}} \in \mathcal{U}$.]



Case 1. Suppose II wins $G_{\mathcal{U}}$ by a strategy τ.
Let player I steal that strategy.
Method of auxiliary games:

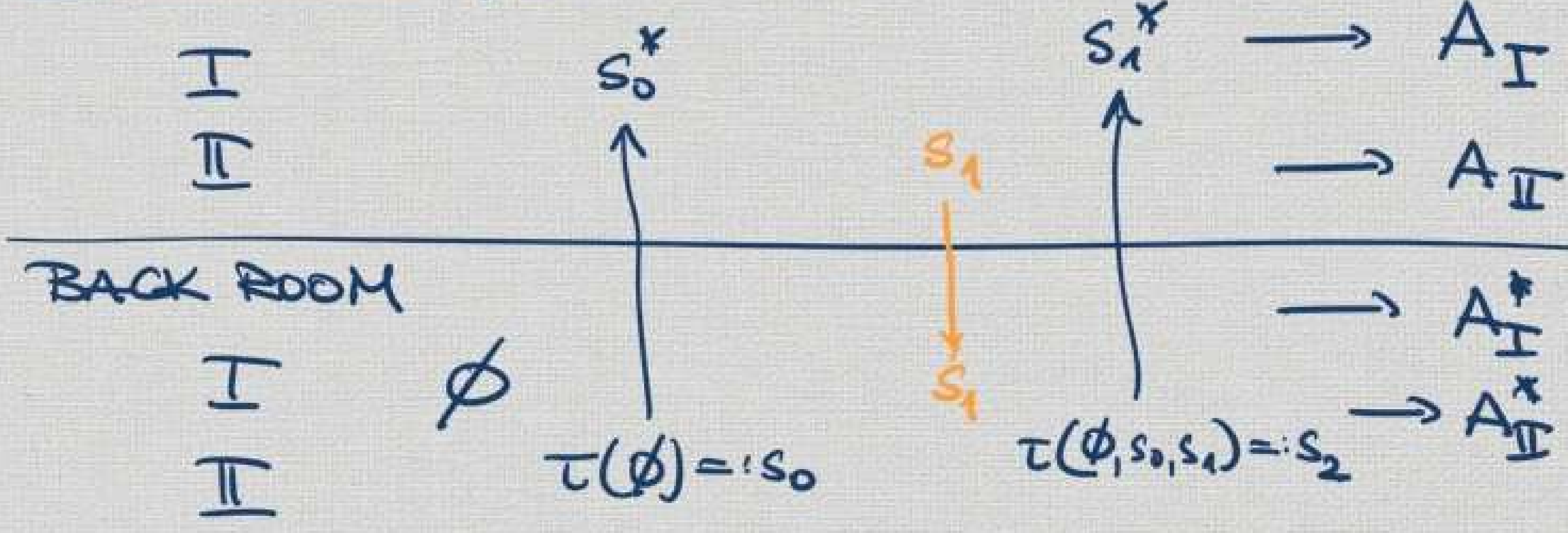


First attempt: $A_{\mathrm{I}} = \bigcup_{i\in\mathbb{N}} s_{2i} = \bigcup_{i\in\mathbb{N}} s_{2i} = A^*_{\mathrm{II}}$

Since τ is winning for II: $A_{\mathrm{II}} = \bigcup_{i\in\mathbb{N}} s_{2i+1} = \emptyset \cup \bigcup_{i\in\mathbb{N}} s_{2i+1} = A^*_{\mathrm{I}}$

$A^*_{\mathrm{I}} \notin \mathcal{U}$. Not in general: $A^*_{\mathrm{II}} \in \mathcal{U}$.

Second attempt



$$s_0^* := s_0 \cup \{0\}$$

$$s_2^* := \begin{cases} s_2 \setminus \{0\} & \text{if } 1 \in s_0 \cup s_1 \\ (s_2 \setminus \{0\}) \cup \{1\} & \text{if } 1 \notin s_0 \cup s_1 \end{cases}$$

$$s_{2n}^* := \begin{cases} s_{2n} \setminus \{0, \dots, n-1\} & \text{if } n \in \bigcup_{k<2n} s_k \\ (s_{2n} \setminus \{0, \dots, n-1\}) \cup \{n\} & \text{if } n \notin \bigcup_{k<2n} s_k . \end{cases}$$

Now $A_{I} \cup A_{II} = \mathbb{N}$,

so $\boxed{A_{I} = \mathbb{N} \setminus A_{II}}$.

Again, since τ was winning, $\boxed{A_{I}^* \notin U}$.

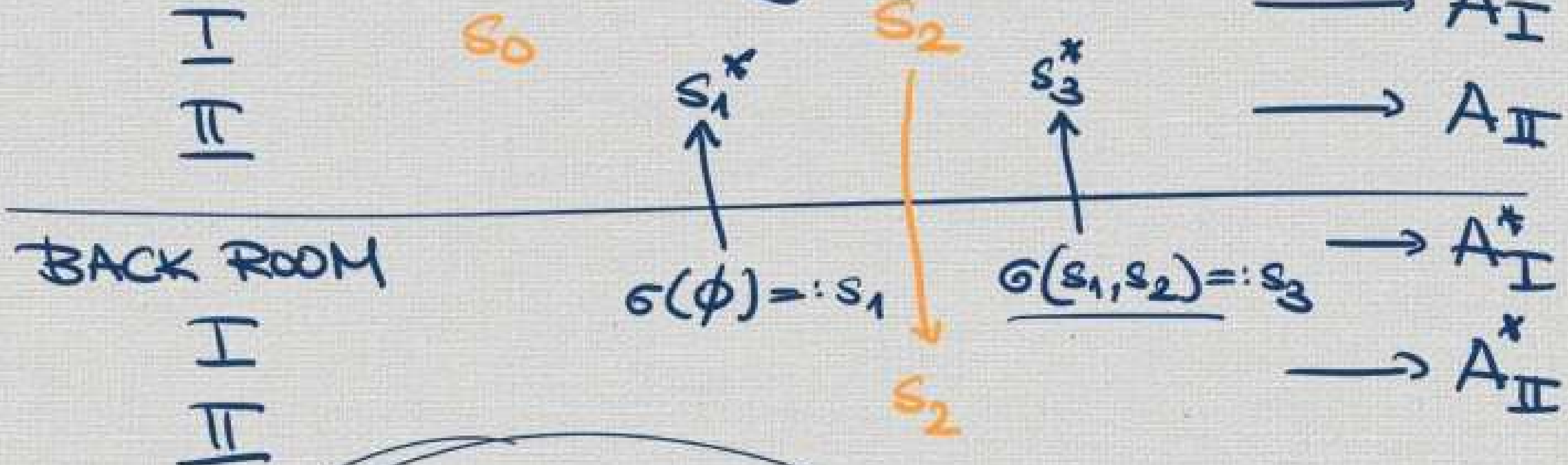
$A_{I}^* \subseteq A_{II}$

$A_{I} \in U$.

Summary. If II has w.s., then we can modify it to a w.s. for I. So: II does not have a w.s.

Case 2. If σ is a w.s. for player I in G_U, construct a w.s. for II.

Once more: auxiliary games.



$$s_1^* := s_1 \setminus s_0$$

$$s_3^* := s_3 \setminus s_0$$

$$s_{2i+1}^* := s_{2i+1} \setminus s_0$$

$$A_{II} = \bigcup_{i \in \mathbb{N}} s_{2i+1}^* = \Big(\bigcup_{i \in \mathbb{N}} s_{2i+1}\Big) \setminus s_0$$

$$= A_I^* \setminus s_0.$$

Since σ is winning, $A_I^* \in U$. By non-principality of U, $A_{II} \in U$. So $A_I \notin U$. Thus II wins in G_U. q.e.d.