

- $G(A)$ $\begin{matrix} \text{I} & x_0 & x_2 & x_4 & \dots \\ \text{II} & & x_1 & x_3 & x_5 & \dots \end{matrix}$

- $G_{aux}(\hat{T})$ $\begin{matrix} \text{I} & \underline{u_0, x_0} & & \underline{u_1, x_2} & & \underline{u_2, x_4} & & \dots \\ \text{II} & & x_1 & & x_3 & & x_5 & \dots \end{matrix}$

Let τ be a winning strategy for player II in $G_{aux}(\hat{T})$.
Fix $s = (x_0, \dots, x_{2n})$, $k_s := |K_s|$, and $\underline{Q} \in [K]^{k_s}$.

Then there is a unique $w_Q : (K_s, <_{K_s}) \rightarrow (Q, <)$
order preserving

Define $\underline{u}^{s,Q}(i) := w_Q \upharpoonright K_{s \uparrow i}$.

$\boxed{S_*^Q} = \underline{u}^{s,Q}(0), x_0, x_1, \dots, \underline{u}^{s,Q}(n), x_{2n}$

$c_s(Q) := \tau(S_*^Q)$

Let \underline{H} [by Rowbottom] be simultaneously homogeneous for all c_s and let $\underline{Q}_{s,H}$ be the first k_s elements of \underline{H} .

$\underline{\tau}_H(s) := \tau(S_*^{Q_{s,H}})$

CLAIM τ_H is a winning strategy for II in $G(A)$.

Suppose not. So there is a σ for player I s.t.

$\underline{x} := \sigma * \tau_H \in A$, $\iff T_x$ is wellfounded

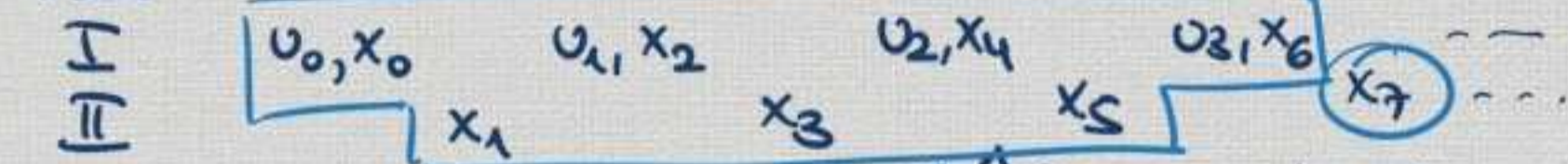
[since \underline{H} is uncountable]

\iff there is a o.p. map $g : (K_x, <_x) \rightarrow (\underline{H}, <)$

Consider

$$v_i := g \uparrow^k x \uparrow^i$$

Then



is a run of $G_{aux}(\hat{T})$ producing $(g, x) \in [\hat{T}]$. In particular, this run is a win for player I in the auxiliary game.

We'll show that this is a win according to the strategy τ . But τ was a w.s. for player II in $G_{aux}(\hat{T})$. Contradiction!

So we need to show that $\tau(S_{*}^{\uparrow}) = x_{2u+1}$ for every u .

Let's write s_{*}^{\uparrow} for $(v_0, x_0, x_1, v_1, x_2, x_3, v_2, \dots, v_u, x_{2u})$
 $S = (x_0, x_1, \dots, x_{2u})$

$$\underline{x_{2u+1}} = \tau_H(x \uparrow^{2u+1}) \quad \text{by choice of } x$$

$$= \tau(S_{*}^{Q_{S,H}}) \quad \text{def of } \tau_H$$

$$= \tau(S_{*}^{\uparrow})$$

by the fact that $\text{ran}(g) \subseteq H$ & H is complete.

QED

Comment

Note that we did not use that H has size κ , only that it is uncountable.

We don't use the full strength of Rowbottom's Theorem, but only

"For every club collection

$$\text{cu. } \bigvee_{\kappa < \omega} [\kappa]^{< \omega} \longrightarrow \omega$$

there is an uncountable simultaneously homogeneous set H ".

Compare: ω_1 -ERDŐS' CARDINALS

This is part of the story why we can't have a converse to Martin's Theorem.

The precise strength of analytic determinacy

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ANALYTIC DETERMINACY AND 0^\sharp

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Martin [12] has shown that the determinacy of analytic games is a consequence of the existence of sharps. Our main result is the converse of this:

THEOREM. *If analytic games are determined, then x^\sharp exists for all reals x .*

This theorem answers question 80 of Friedman [5]. We actually obtain a somewhat sharper result; see Theorem 4.1. Martin had previously deduced the existence of sharps from Σ^1_1 -determinacy (where Σ^1_α is the α th level of the difference hierarchy based on Σ^1_1 ; see [1]). Martin has also shown that the existence of sharps implies $< \omega^2$ - Σ^1_1 -determinacy.

ZERO SHARP

- Zero Sharp is an unusual large cardinal axiom which doesn't talk about large cardinals, but rather about a real number which codes truth functions for inner models.
- For large cardinal notions of this type, we can have real equivalence.

Then (ZFC)

$\text{Det}(\Sigma^1_1) \iff$ for all $x \in \omega^\omega$, the object x^\sharp exists.

Our final goal in this course:

Theorem (Solovay).

If AD holds, then \aleph_1 is a measurable cardinal.

[This implies that there is an inner model M s.t. $M \models ZFC + \text{there is a measurable cardinal.}$

This requires techniques we didn't do in this course.]

NOTE "being measurable" only implies being large if it implies inaccessibility. But remember the proof of "meas. \Rightarrow inacc." This required that 2^{\aleph_1} $[1 < \kappa]$ is wellorderable.

In our setting above 2^{\aleph_0} is clearly not wellorderable.

Working in ZF + AD means that we have to be very cautious about almost everything we've done so far.

The most important fragment of AC that we used in our account of descriptive set theory was

$$AC_{\omega}(\mathbb{R}) \Leftrightarrow AC_{\omega}(\omega^{\omega})$$

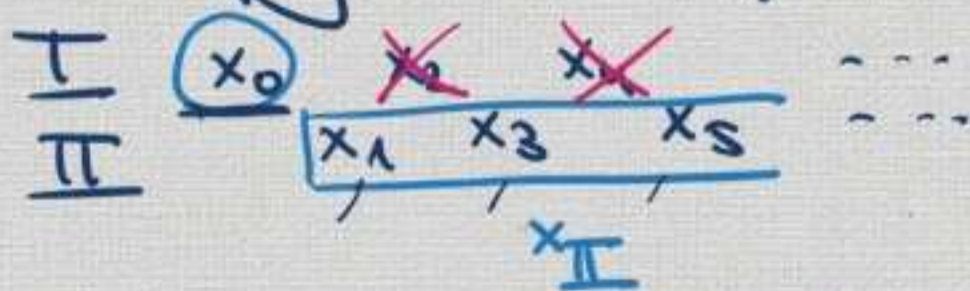
In fact, everything we've done on Borel and projective sets (with exception of Martin's Borel det.) can be done in $ZF + AC_{\omega}(\mathbb{R})$.

ES#1 (4) $ZF + AC_{\omega}(\mathbb{R}) \implies$

\aleph_1 is regular.
and the proof of $\sum_{\alpha < \omega_1} \omega_{\alpha} = \prod_{\alpha < \omega_1} \omega_{\alpha}$ used exactly that.

Proposition $AD \implies AC_{\omega}(\omega^{\omega})$.

Proof. Suppose $(X_n; n \in \mathbb{N})$ is a seq. of non-empty subsets of ω^{ω} .



Player II wins
if $x_{II} \in X_{x_0}$

Player I cannot have a w.s. A w.s. for player I is essentially a natural number x_0 . If $y \in X_{x_0}$ and τ_y is the blindfolded ch. "play y ", then $x_0 * \tau_y$ is a win for II.

So, by AD, player II has a w.s. τ .

Consider $(k * \tau)_{\underline{\Pi}} \in X_k$

So $k \mapsto (k * \tau)_{\underline{\Pi}}$ is
a choice fn for $(X_u; u \in \mathbb{N})$.
q.e.d.

Remark This proof generalises to

$AD_M \implies AC_M(M^\omega)$.

\Downarrow
for all $A \subseteq M^\omega$, the game
 $G(A)$ is determined

In particular, $AD_{\mathbb{R}} \implies AC_{\mathbb{R}}(\mathbb{R})$

Uniformisation principle
 $\exists S \# 1(S)$.

What about ultrafilters?

If X is any set and $a \in X$, then

$$\{A \subseteq X; a \in A\}$$

is a principal (and therefore κ -complete for all κ) ultrafilter. (ZF, no choice)

But the proof that arbitrary filters on X can be extended to an ultrafilter is an application of Zorn's Lemma, so not in general a ZF-theorem.

Q. Is there a non-principal ultrafilter on \mathbb{N} ?

Theorem AD \implies there are no non-principal ultrafilters on \mathbb{N} . (*)

This implies that every ultrafilter is \aleph_1 -complete. (**)

So, we're left with showing that there is a non-principal uf. on \aleph_1 .