XXI Twenty-first Lecture

INFINITE GAMES

LENT 2021

10 MARCH 2021

- $G(A)$

$$\begin{array}{llllllll} \mathrm{I} & x_0 & & x_2 & & x_4 & & \cdots \\ \mathrm{II} & & x_1 & & x_3 & & x_5 & \cdots \end{array}$$

- $G_{aux}(\hat{T})$

$$\begin{array}{llllllll} \mathrm{I} & u_0, x_0 & & u_1, x_2 & & u_2, x_4 & & \cdots \\ \mathrm{II} & & x_1 & & x_3 & & x_5 & \cdots \end{array}$$

Let τ be a winning strategy for player II in $G_{aux}(\hat{T})$.

Fix $s = (x_0, \dots, x_{2n})$, $k_s := |K_s|$, and $Q \in [\kappa]^{k_s}$.

Then there is a unique $w_Q : (K_s, <_{KB}) \longrightarrow (Q, <)$ order preserving

Define $u^{s,Q}(i) := w_Q \upharpoonright K_{s \upharpoonright i}$.

$$\boxed{s_*^Q} = u^{s,Q}(0), x_0, x_1, \dots, u^{s,Q}(n), x_{2n}$$

$$c_s(Q) := \tau(s_*^Q)$$

Let H [by Rowbottom] be simultaneously homogeneous for all c_s and let $Q_{s,H}$ be the first k_s elements of H.

$$\tau_H(s) := \tau(s_*^{Q_{s,H}})$$

CLAIM τ_H is a winning strategy for II in $G(A)$.

Suppose not. So there is a σ for player I s.t.

$x := \sigma * \tau_H \in A$. $\iff$ T_x is wellfounded

$\iff$ there is a o.p. map

$g : (K_x, <_x) \longrightarrow (H, <)$

[since H is uncountable]

Consider

$$v_i := g \upharpoonright K_{x \upharpoonright i}$$

Then

I	v_0, x_0		v_1, x_2		v_2, x_4		v_3, x_6	$\cdots$
II		x_1		x_3		x_5		x_7 $\cdots$

is a run of $G_{aux}(\hat{T})$ producing $(g, x) \in [\hat{T}]$. In particular, this run is a win for player I in the auxiliary game.

We'll show that this is a run according to the strategy τ. But τ was a w.s. for player II in $G_{aux}(\hat{T})$. Contradiction!

So we need to show that $\tau(s_*^g) = x_{2n+1}$ for every n.

[Let's write s_*^g for $(v_0, x_0, x_1, v_1, x_2, x_3, v_2, \dots, v_n, x_{2n})$
$s = (x_0, x_1, \dots, x_{2n})$]

$$\underline{x_{2n+1}} = \tau_H(\underbrace{x \upharpoonright 2n+1}_{s})$$ by choice of x

$$= \tau\left(s_*^{Q_{s,H}}\right)$$ def of τ_H

$$= \tau\left(s_*^g\right)$$ by the fact that $\mathrm{ran}(g) \subseteq H$ & H is homogen.

QED

Comment

Note that we did not need that H has size κ, only that it is uncountable.

We don't need the full strength of Rowbottom's Thm, but only

"For every ctble collection
$$c_n : [\kappa]^{<\omega} \longrightarrow \omega$$
there is an uncountable simultaneously homogeneous set H".

Compare : ω_1-ERDŐS CARDINALS

This is part of the story why we can't have a converse to Martin's Theorem.

The precise strength of analytic determinacy

THE JOURNAL OF SYMBOLIC LOGIC
Volume 43, Number 4, Dec. 1978

ZERO SHARP

ANALYTIC DETERMINACY AND $0^\#$

LEO HARRINGTON[1]

Martin [12] has shown that the determinacy of analytic games is a consequence of the existence of sharps. Our main result is the converse of this:

THEOREM. *If analytic games are determined, then $x^\#$ exists for all reals x.*

This theorem answers question 80 of Friedman [5]. We actually obtain a somewhat sharper result; see Theorem 4.1. Martin had previously deduced the existence of sharps from $3 - \boldsymbol{\Pi}^1_1$-determinacy (where $\alpha - \boldsymbol{\Pi}^1_1$ is the αth level of the difference hierarchy based on $\boldsymbol{\Pi}^1_1$; see [1]). Martin has also shown that the existence of sharps implies $< \omega^2 - \boldsymbol{\Pi}^1_1$-determinacy.

- Zero Sharp is an unusual large cardinal axiom which doesn't talk about large cardinals, but rather about a real number which codes truth functions for inner models.
- For large cardinal notions of this type, we can have real equivalence.

Thm (ZFC)

$$\mathrm{Det}(\boldsymbol{\Sigma}^1_1) \iff \text{for all } x \in \omega^\omega, \text{ the object } x^\# \text{ exists.}$$

Our final goal in this course:

Theorem (Solovay).

If AD holds, then $\aleph_1$ is a measurable cardinal.

[This implies that there is an inner model M s.t. $M \models ZFC$ + there is a measurable cardinal.

This requires techniques we didn't do in this course.]

NOTE "being measurable" only implies being large if it implies inaccessibility. But remember the proof of "meas. $\Rightarrow$ inacc" This required that 2^λ $[\lambda < \kappa]$ is wellorderable.

In our setting above $2^{\aleph_0}$ is clearly not wellorderable.

Working in ZF + AD means that we have to be very cautious about almost everything we've done so far.

The most important fragment of AC that we used in our account of descriptive set theory was

$$AC_\omega(\mathbb{R}) \iff AC_\omega(\omega^\omega)$$

In fact, everything we've done on Borel and projective sets [with exception of Martin's Borel det.] can be done in $ZF + AC_\omega(\mathbb{R})$.

ES#1 (4) $ZF + AC_\omega(\mathbb{R}) \Longrightarrow \aleph_1$ is regular.

and the proof of $\underset{\sim}{\Sigma}^0_{\omega_1} = \underset{\sim}{\Pi}^0_{\omega_1}$ used exactly that.

<u>Proposition</u> $AD \Longrightarrow AC_\omega(\omega^\omega)$.

<u>Proof.</u> Suppose $(X_n \,;\, n \in \mathbb{N})$ is a seq. of non-empty subsets of ω^ω.

$$\begin{array}{lllll} \text{I} & x_0 & \cancel{x_2} & \cancel{x_4} & \cdots \\ \text{II} & \underbrace{x_1 \quad x_3 \quad x_5}_{x_{\text{II}}} & & & \cdots \end{array}$$

Player II wins if $x_{\text{II}} \in X_{x_0}$

Player I cannot have a w.s. A w.s. for player I is essentially a natural number x_0. If $y \in X_{x_0}$ and τ_y is the blindfolded str. "play y", then $x_0 * \tau_y$ is a win for II.

So, by AD, player II has a w.s. τ.

Consider $(k * \tau)_{II} \in X_k$

So $k \longmapsto (k * \tau)_{II}$ is a choice fn for $(X_n ; n \in \mathbb{N})$.

q.e.d.

<u>Remark</u> This proof generalizes to

$$AD_M \Longrightarrow AC_M(M^\omega).$$

$\Updownarrow$

for all $A \subseteq M^\omega$, the game $G(A)$ is determined

In particular, $AD_{\mathbb{R}} \Longrightarrow \boxed{AC_{\mathbb{R}}(\mathbb{R})}$

Uniformization principle ES #1 (5).

What about ultrafilters?

If X is any set and $a \in X$, then
$$\{A \subseteq X \,;\, a \in A\}$$
is a principal (and therefore κ-complete for all κ) ultrafilter. (ZF, no choice)

But the proof that arbitrary filters on X can be extended to an ultrafilter is an application of Zorn's Lemma, so not in general a ZF-theorem.

Q. Is there a non-principal ultrafilter on $\mathbb{N}$?

Theorem AD $\Longrightarrow$ there are no non-principal ultrafilters on $\mathbb{N}$.
(*) (**)

This implies that every ultrafilter is $\aleph_1$-complete.

So, we're left with showing that there is a nonprincipal uf. on $\aleph_1$.