XX TWENTIETH LECTURE

LENT 2021
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INFINITE GAMES

Theorem 1 (Shoenfield). If κ is uncountable, then every $\mathbf{\Pi}^1_1$ set is κ-Suslin.

Notation. Fix a bijection $i \mapsto s_i$ from $\omega \to \omega^{<\omega}$ such that if $s_i \subseteq s_j$, then $i \leq j$. (This implies that $\mathrm{lh}(s_i) \leq i$.) Let $T \subseteq (\omega \times \omega)^{<\omega}$ be a tree, $x \in \omega^\omega$, and $s \in \omega^{<\omega}$. Then we let

M is the set of partial functions from ω to κ with finite domain.

Need bij. $M \longleftrightarrow \kappa$.

A $\underset{\sim}{\Pi}^1_1$ s.t. $x \in A$ iff T_x is wellfounded

$$T_s := \{t \in \omega^{<\omega};\ (s\restriction \mathrm{lh}(t), t) \in T\},$$

$$T_x := \{t \in \omega^{<\omega};\ (x\restriction \mathrm{lh}(t), t) \in T\} = \bigcup_{n\in\mathbb{N}} T_{x\restriction n},$$

$$K_s := \{i \leq \mathrm{lh}(s);\ s_i \in T_s\}, \text{ and}$$

$$K_x := \{i \in \omega;\ s_i \in T_x\} = \bigcup K_{x\restriction n}.$$

If $s \in \omega^{<\omega}$ and $u \in M^{<\omega}$ such that $\mathrm{lh}(u) \leq \mathrm{lh}(s)$, we say that u is *coherent with* s if

- (1) for all $i < \mathrm{lh}(u)$, we have that $\mathrm{dom}(u(i)) = K_{s\restriction i}$,
- (2) for all $i < \mathrm{lh}(u)$, $u(i)$ is an order preserving map from $(K_{s\restriction i}, <_{\mathrm{KB}})$ into $(\kappa, <)$, and
- (3) for $i \leq j$, we have that $u(i) \subseteq u(j)$.

We now define the *Shoenfield tree* on $\overset{M}{\cancel{\omega}} \times \overset{\omega}{\cancel{M}}$ by $\hat{T} := \{(\overset{u,s}{\cancel{s}, \cancel{u}});\ u \text{ is coherent with } s\}$

1.

CLAIM: $A = p[\hat{T}]$

Already done: $A \subseteq p[\hat{T}]$ Lecture XIX

Remarks: $\boxed{p[\hat{T}] \subseteq A.}$

Proof of 2nd inclusion. $x \in p[\hat{T}]$

$\iff$ there is v s.t. $(v, x) \in [\hat{T}]$

$\iff$ there is v s.t. f.a. n $(v\restriction n, x\restriction n) \in \hat{T}$

$\iff$ there is v s.t. f.a. n $v\restriction n$ is coherent with $x\restriction n$.

(1)/(2) $v(i) : K_{x\restriction i} \longrightarrow \kappa$ o.p. (3) $v(i) \subseteq v(j)$ for $i \leq j$.

By (3), define $\hat{v} : K_x \longrightarrow \kappa$ by $\hat{v} := \bigcup_{i\in\mathbb{N}} v(i)$.

If $\hat{\sigma}$ is orderpreserving from

$(K_x, <_{KB})$ into $(\kappa, <)$,

then T_x is wellfounded and thus $x \in A$.

So, we're left to show that $\hat{\sigma}$ is o.p.

Suppose not:

then there are $i, j \in K_x$ s.t.

$s_i <_{KB} s_j$ but $\boxed{\hat{\sigma}(i) \not< \hat{\sigma}(j)}$

[Since $K_x = \bigcup_{n \in \mathbb{N}} K_{x \upharpoonright n}$, there is $n \in \mathbb{N}$ s.t.

$i, j \in K_{x \upharpoonright n}$.]

But $\sigma(n)$ was orderpreserving:

$s_i <_{KB} s_j \iff \boxed{\sigma(n)(i) < \sigma(n)(j)}$

This is a contradiction!

Thus $\hat{\sigma}$ is o.p.; thus $x \in A$.

q.e.d.

Theorem (Martin, 1969/70).

If there is a measurable cardinal, then all $\underset{\sim}{\Pi}^1_1$ sets are determined.

Def. κ is measurable $\iff$ there is U κ-complete, non-principal on κ

Theorem (ZFC, no proof)

If κ is measurable, then there is a normal κ-complete non-principal uf. on κ.

Theorem (Rowbottom, ES #3)

If U is normal on κ and $\gamma < \kappa$ and $c_n : [\kappa]^{<\omega} \longrightarrow \gamma$, then there is a set $H \in U$ s.t. for all $n, k \in \mathbb{N}$

$$c_n \upharpoonright [H]^k \text{ is constant.}$$

We say κ satisfies Rowbottom's Thm if for all $\gamma < \kappa$, $c_n : [\kappa]^{<\omega} \longrightarrow \gamma$ there is H with $|H| = \kappa$ s.t. for all $n, k \in \mathbb{N}$ $c_n \upharpoonright [H]^k$ is constant.

SUMMARY ZFC implies that measurable cardinals satisfy Rowbottom's Thm.

Theorem (Martin). If κ satisfies Rowbottom's Theorem, then all $\underset{\sim}{\Pi}^1_1$ sets are determined.

PROOF. Fix $A \in \underset{\sim}{\Pi}^1_1$. By Shoenfield, we have $\hat{T}$ tree on $\kappa \times \omega$ s.t.

$$A = p[\hat{T}].$$

We know that if you compare $G(A)$ and $G_{aux}(\hat{T})$, we get:

- $G_{aux}(\hat{T})$ is determined [Gale-Stewart]
- w.s. of player I transfers from $G_{aux}(\hat{T})$ to $G(A)$

So we still need:

$(*)$ $\boxed{\begin{array}{l}\text{If player II has a w.s. in } G_{aux}(\hat{T}), \\ \text{then player II has a w.s. in } G(A).\end{array}}$

$$\begin{array}{lcccccccc} \mathrm{I} & x_0 & & x_2 & & x_4 & \ldots & & x \in A \\ \mathrm{II} & & x_1 & & x_3 & & & & \end{array}$$

$$\begin{array}{lcccccccc} \mathrm{I} & u_0, x_0 & & u_1, x_2 & & u_2, x_4 & & \ldots & (u,x) \in [\hat{T}] \\ \mathrm{II} & & x_1 & & x_3 & & x_5 & & \end{array}$$

τ

$$\begin{array}{llllll} \mathrm{I} & v_0, x_0 & & v_1, x_2 & & \overset{?}{\bullet}, x_4 \\ \mathrm{II} & & x_1 & & x_3 & \end{array}$$

Can assume that v_0 is player s.t. player I does not lose immediately.

So: $v_0 : K_{x\restriction 0} \longrightarrow K$ order preserving.

$$K_{x\restriction 0} = K_{\emptyset} = \{ i \leq 0 ; s_i \in T_{x\restriction 0} \} = \{0\}$$

One step further

$$v_1 : K_{x\restriction 1} \longrightarrow K$$

$$K_{x\restriction 1} = K_{(x_0)} = \{ i \leq 1 ; s_i \in T_{x\restriction 1} \} = \begin{cases} \{0\} \\ \underbrace{\{0,1\}}_{\iff s_1 \in T_{x_0}} \end{cases}$$

either a single elt. of K or a two-element subset of K.

Let $k_S := |K_S|$ and $Q \in [k]^{k_S}$. Then Q determines a unique order preserving map

$$w_Q : (\underline{K_S, <_{KB}}) \longrightarrow (Q, <).$$

Define

$$v^{S,Q}(i) := w \upharpoonright K_{S \upharpoonright i}.$$

Then $v^{S,Q}$ is coherent with S. This allows me to translate a position in the original game

$$\begin{array}{lccccc} \mathrm{I} & x_0 & & x_2 & \cdots & x_{2n} \\ \mathrm{II} & & x_1 & & x_3 \quad \cdots & \end{array} \longleftarrow S$$

into a (legal) position in the auxiliary game

$$\begin{array}{lccccc} \mathrm{I} & v^{S,Q}(0), x_0 & & v^{S,Q}(1), x_2 & \cdots & v^{S,Q}(n), x_{2n} \\ \mathrm{II} & & x_1 & & x_3 \quad \cdots & \end{array}$$

$\uparrow$ S_*^Q Position in $M^{<\omega} \times \omega^{<\omega}$

Define a colouring $s \in \omega^{<\omega}$.

$$c_s : [\kappa]^{k_s} \longrightarrow \omega$$

by $$c_s(Q) := \underline{\tau(s_*^Q)}.$$

$Q \in [\kappa]^{k_s}$

where τ is our w.s. for player $\underline{\text{II}}$ in $G_{\infty x}(\hat{T})$.

This is a family of countably many colourings, so by Rowbottom, we find $H \subseteq \kappa$, $|H| = \kappa$ s.t.

$$c_s \restriction [H]^{k_s} \text{ is constant.}$$

[i.e., $Q, Q' \in [H]^{k_s}$, then $c_s(Q) = c_s(Q')$,

$$\text{so} \quad \tau(\underline{s_*^Q}) = \tau(\underline{s_*^{Q'}}).$$

<u>In words</u>: If II fills in the gaps by <u>any</u> $Q \in [H]^{k_s}$, then the answer of τ does not depend on the precise choice of Q.

So, let

$Q_{H,s}$:= the first k_s many elements of H.

Define strategy τ_H in the game $G(A)$ for player II:

$$\tau_H(s) := \tau\left(s_*^{Q_{H,s}}\right)$$

I	$u^{s,Q_{H,s}(0)}, x_0$		$u^{s,Q_{H,s}(1)}, x_2$		$u^{s,Q_{H,s}(2)}, x_4$		...
II		x_1		x_3		x_5	...

$s_*^{Q_{H,s}}$

<u>CLAIM</u>: If τ was winning in $G_{aux}(\hat{T})$, then τ_H is winning in $G(A)$.