Theorem 1 (Shoenfield). If $\kappa$ is uncountable, then every $\Pi^1_1$ set is $\kappa$-Suslin.

Notation. Fix a bijection $i \mapsto s_i$ from $\omega \to \omega^{<\omega}$ such that if $s_i \subseteq s_j$, then $i \leq j$. (This implies that $\lh(s_i) \leq i$.) Let $T \subseteq (\omega \times \omega)^{<\omega}$ be a tree, $x \in \omega^\omega$, and $s \in \omega^{<\omega}$. Then we let

$$T_s := \{ t \in \omega^{<\omega} ; (s \upharpoonright \lh(t), t) \in T \},$$

$$T_x := \{ t \in \omega^{<\omega} ; (x \upharpoonright \lh(t), t) \in T \} = \bigcup_{n \in \omega} T_{x|n},$$

$$K_s := \{ i \in \omega ; s_i \in T_s \},$$

$$K_x := \{ i \in \omega ; s_i \in T_x \} = \bigcup_{n \in \omega} K_{x|n}.$$

If $s \in \omega^{<\omega}$ and $u \in M^{<\omega}$ such that $\lh(u) \leq \lh(s)$, we say that $u$ is coherent with $s$.

We now define the Shoenfield tree on $\mathcal{M} \times \mathcal{M}$ by $\hat{T} := \{ (u, s) ; u \text{ is coherent with } s \}$.

**Claim:** $A = p[\hat{T}]$

**Already done:** $A \subseteq p[\hat{T}]$

**Remains:** $p[\hat{T}] \subseteq A$

**Proof of 2nd inclusion.** $x \in p[\hat{T}]$

$$\iff \text{there is } u \text{ s.t. } (u, x) \in [\hat{T}]$$

$$\iff \text{there is } u \text{ s.t. f.a.n. } (\cup u, x) \upharpoonright n \in [\hat{T}]$$

$$\iff \text{there is } u \text{ s.t. f.a.n. } u \text{ is coherent with } x \text{ n.d. }$$

(1) for $i \leq j$, define $\hat{u}(i) : K_x \to K_{u \cup u(i)}$ by

(2) $u(i) : K_x \to K_{u \cup u(i)}$

by (3), define $\hat{u} : K_x \to K_{u \cup u(i)}$ for $i \leq j$. 

By (3), define $\hat{u} : K_x \to K_{u \cup u(i)}$. 

If $\delta$ is order-preserving from $(K_X, <_{KB})$ into $(K, <)$, then $T_\delta$ is well-founded and thus $x \in A$. So, we're left to show that $\delta$ is o.p.

Suppose not:

Then there are $i, j \in K_X$ s.t.

$s_i <_{KB} s_j$ but $\delta(i) \neq \delta(j)$

[Since $K_X = \bigcup_{n \in \mathbb{N}} K_{X \times N}$, there is $n \in \mathbb{N}$ s.t.

$i, j \in K_{X \times N}$]

But $u(n)$ was order-preserving:

$s_i <_{KB} s_j$ but $u(n)(i) = u(n)(j)$

This is a contradiction!

Thus $\delta$ is o.p.; thus $x \in A$. q.e.d.
Theorem (Martin, 1969/70).

If there is a measurable cardinal, then all \( \mathcal{P}(\kappa) \) sets are determined.

**Def.** \( \kappa \) is measurable \( \iff \)

\( \kappa \) is an \( \mathcal{U} \) k-complete, non-principal on \( \kappa \).

**Theorem** (ZFC, no proof)

If \( \kappa \) is measurable, then there is a normal k-complete non-principal ultrafilter on \( \kappa \).

**Theorem** (Rowbottom, ES #3)

If \( \mathcal{U} \) is normal on \( \kappa \) and \( \mathcal{F} \leq \kappa \) and \( \kappa : [\kappa]^{<\omega} \rightarrow \mathcal{F} \), then there is a set \( \mathcal{H} \subseteq \mathcal{U} \) s.t. for all \( n, k \in \mathbb{N} \)

\( c_n \bigcup [\mathcal{H}]^k \) is constant.

We say \( \kappa \) satisfies Rowbottom’s Theorem if for all \( \mathcal{F} \)

\( \forall \kappa, \mathcal{F} : [\kappa]^{<\omega} \rightarrow \mathcal{F} \) s.t. \( \mathcal{F} \subseteq \mathcal{H} \) with \( |\mathcal{H}| = \kappa \)

satisfy Rowbottom’s Theorem.

**Summary** ZFC implies that measurable cardinals satisfy Rowbottom’s Theorem.
Prove (Matrix). If $k$ satisfies Rowbottom's

| Prove, then all $\Pi^1_1$ sets are determined. |

**Proof.** Fix $A \in \Pi^1_1$. By Shoenfield, we have a tree on $\mathbb{N} \times \omega$ such that

$$A = p[\uparrow]$$

We know that if you compare $G(A)$ and $G_{\omega \times} (\uparrow)$, we get:

- $G_{\omega \times} (\uparrow)$ is determined
- Gale-Stewart

- w.s. of player I transfers from $G_{\omega \times} (\uparrow)$ to $G(A)$

So we still need:

$$\text{(x)}$$

- If player II has a w.s. in $G_{\omega \times} (\uparrow)$
- Then player II has a w.s. in $G(A)$.

\[
\begin{array}{cccccccccc}
I & II & x_0 & x_1 & x_2 & x_3 & x_4 & \ldots & x \in A \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
I & II & u_0, x_0 & u_1, x_2 & x_3 & \ldots & (u, x) \in \mathcal{F} \\
\end{array}
\]
Can assume that $u_0$ is player s.t. player I does not lose immediately.

So: $u_0 : K^{\times 10} \rightarrow K$ order preserving.

One step further:

$u_1 : K^{\times M} \rightarrow K$

either a single elt. of $K$ or a two-element subset of $K$. 

\[
\begin{align*}
\{ i \leq 0 \} & \subseteq T^{\times 10} \\
\{ i \leq 1 \} & \subseteq T^{\times M} \\
\{ s \} & \subseteq T_{x_0} \\
\{ (0,1) \} & \subseteq T_{x_0} \\
\end{align*}
\]
Let $k_s := |K_s|$ and $Q \in [k]^{k_s}$. Then $Q$ determines a unique order preserving map

$$w_Q : (K_s, \prec_{K_s}) \rightarrow (Q, \prec)$$

Define $u_{s,Q}(i) := w \sqrt{K_s i}$. Then $u_{s,Q}$ is coherent with $s$. This allows me to translate a position in the original game $I$ into a (legal) position in the auxiliary game $S$.

$$u_{s,Q}(0), x_0 \quad u_{s,Q}(1), x_1 \quad u_{s,Q}(2), x_2 \quad \ldots \quad u_{s,Q}(n), x_n$$

Position $m$ $M < \omega \times \omega < \omega$
Define a colouring

\[ c_s : [k]^k \rightarrow \omega \]

by

\[ c_s(Q) = \tau(s^*) \]

where \( \tau \) is our c.w.s. for player II in \( G_{\omega \times \{A\}} \).

This is a family of countably many colourings, so by Rowbottom, we find \( H \subseteq k \) s.t.

\[ c_s[H]^k \] is constant.

[i.e., \( Q, Q' \in [H]^k \), then \( c_s(Q) = c_s(Q') \),

so \( \tau(s^*) = \tau(s^{*'}) \).

In words: if II fills in the gaps by any \( Q \in [H]^k \), then the answer of \( \tau \)
does not depend on the precise choice of \( Q \).]
So, let
\[ Q_{H, S} := \text{the first } k \text{ many elements of } H. \]

Define strategy \( T_H \) in the game \( G(A) \) for player II:
\[ T_H(s) := \tau(s_{Q_{H, S}(s)}). \]

CLAIM: If \( T \) was winning in \( G_{00x}^{(1)} \), then \( T_H \) is winning in \( G(A) \).