

TWENTIETH LECTURE

INFINITE GAMES

LENT 2021
8 March 2021

Theorem 1 (Shoenfield). If κ is uncountable, then every Π_1^1 set is κ -Suslin.

Notation. Fix a bijection $i \mapsto s_i$ from $\omega \rightarrow \omega^{<\omega}$ such that if $s_i \subseteq s_j$, then $i \leq j$. (This implies that $\text{lh}(s_i) \leq i$.) Let $T \subseteq (\omega \times \omega)^{<\omega}$ be a tree, $x \in \omega^\omega$, and $s \in \omega^{<\omega}$. Then we let

M is the set of partial functions from ω to κ with finite domain.

$$T_s := \{t \in \omega^{<\omega}; (s \upharpoonright \text{lh}(t), t) \in T\},$$
$$T_x := \{t \in \omega^{<\omega}; (x \upharpoonright \text{lh}(t), t) \in T\} = \bigcup_{n \in \mathbb{N}} T_{x \upharpoonright n},$$
$$K_s := \{i \leq \text{lh}(s); s_i \in T_s\}, \text{ and}$$
$$K_x := \{i \in \omega; s_i \in T_x\} = \bigcup_{n \in \mathbb{N}} K_{x \upharpoonright n}.$$

$A \Pi_1^1$ s.t.
 $x \in A$ iff \hat{T}_x
is well founded.

Need bij. $M \leftrightarrow \kappa$.

If $s \in \omega^{<\omega}$ and $u \in M^{<\omega}$ such that $\text{lh}(u) \leq \text{lh}(s)$, we say that u is coherent with s if

- (1) for all $i < \text{lh}(u)$, we have that $\text{dom}(u(i)) = K_{s \upharpoonright i}$,
- (2) for all $i < \text{lh}(u)$, $u(i)$ is an order preserving map from $(K_{s \upharpoonright i}, <_{KB})$ into $(\kappa, <)$, and
- (3) for $i \leq j$, we have that $u(i) \subseteq u(j)$.

We now define the Shoenfield tree on $M \times \omega$ by $\hat{T} := \{(u, s); u \text{ is coherent with } s\}$

CLAIM :

$$A = p[\hat{T}]$$

Already done:

$$A \subseteq p[\hat{T}]$$

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Remains :

$$p[\hat{T}] \subseteq A.$$

Proof of 2nd inclusion. $x \in p[\hat{T}]$

\iff there is u s.t. $(u, x) \in \hat{T}$

\iff there is u s.t. f.a. n $(u \upharpoonright n, x \upharpoonright n) \in \hat{T}$

\iff there is u s.t. f.a. n $u \upharpoonright n$ is coherent with $x \upharpoonright n$.

(1)/(2) $u(i) : K_{x \upharpoonright i} \rightarrow \kappa$ o.p. (3) $u(i) \subseteq u(j)$ for $i \leq j$.

By (3), define $\hat{u} : K_x \rightarrow \kappa$
by $\hat{u} := \bigcup_{i \in \mathbb{N}} u(i)$.

If $\hat{\sigma}$ is order preserving from

$(K_x, <_{KB})$ into $(K, <)$,

then T_x is wellfounded and thus $x \in A$.

So, we're left to show that $\hat{\sigma}$ is o.p.

Suppose not:

then there are $i, j \in K_x$ s.t.

$s_i <_{KB} s_j$

but

$\hat{\sigma}(i) \not< \hat{\sigma}(j)$

[Since $K_x = \bigcup_{n \in \mathbb{N}} K_x \upharpoonright_n$, there is $n \in \mathbb{N}$ s.t.

$i, j \in K_x \upharpoonright_n$.

But $\sigma(n)$ was order preserving:

$s_i <_{KB} s_j \iff$

$\sigma(n)(i) < \sigma(n)(j)$

This is a contradiction!

Thus $\hat{\sigma}$ is o.p.; thus $x \in A$.

q.e.d.

Theorem (Martin, 1969/70).

If there is a measurable cardinal, then all \aleph_1 sets are determined.

Def. κ is measurable \iff
there is \cup κ -complete, non-principal
on κ

Theorem (ZFC, no proof)

If κ is measurable, there is a
normal κ -complete non-principal of. on κ .

Theorem (Rowbottom, ES # 3)

If \cup is normal on κ and $\gamma < \kappa$ and
 $c_u: [\kappa]^{<\omega} \rightarrow \gamma$, there is
a set $H \in \cup$ s.t. for all $n, k \in \mathbb{N}$
 $c_u \upharpoonright [H]^k$ is constant.

We say κ satisfies Rowbottom's Theorem if for all
 $\gamma < \kappa$, $c_u: [\kappa]^{<\omega} \rightarrow \gamma$ there is H with $|H| = \kappa$
s.t. for all $n, k \in \mathbb{N}$ $c_u \upharpoonright [H]^k$ is constant.

SUMMARY ZFC implies that measurable
cardinals satisfy Rowbottom's Theorem.

Reverse (Martin). If K satisfies Rowbottom's Reverse, then all $\mathbb{N}^{\mathbb{N}}$ sets are determined.

PROOF. Fix $A \in \mathbb{N}^{\mathbb{N}}$. By Silverfield, we have \hat{t} tree on $M \times \omega$ s.t.

$$A = p[\hat{t}].$$

We know that if you compare $G(A)$ and $G_{\text{aux}}(\hat{t})$, we get:

- $G_{\text{aux}}(\hat{t})$ is determined [Gale-Stewart]
- w.s. of player I transfer from $G_{\text{aux}}(\hat{t})$ to $G(A)$

So we still need:

(*) If player II has a w.s. in $G_{\text{aux}}(\hat{t})$, then player II has a w.s. in $G(A)$.

• II x_0 x_1 x_2 x_3 x_4 ... $x \in A$

• II u_0, x_0 x_1 u_1, x_2 x_3 u_2, x_4 x_5 ... $(u, x) \in [\hat{t}]$



Can assume that u_0 is player s.t. player I does not lose immediately.

So: $u_0: K_{x \times 10} \longrightarrow K$ order preserving.

$$\begin{aligned}
 &= K_{x \times 10} \\
 &= \emptyset \\
 &= \{i \leq 0; s_i \in T_{x \times 10}\} \\
 &= \{0\}
 \end{aligned}$$

One step further

$$u_1: K_{x \times 11} \longrightarrow K$$

either a single elt. of K or a two-element subset of K .

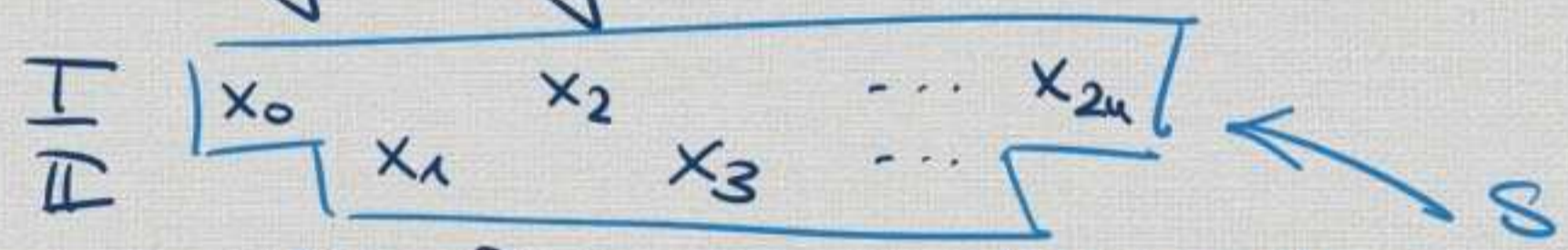
$$\begin{aligned}
 &= K_{(x_0)} \\
 &= \{i \leq 1; s_i \in T_{x \times 11}\} = \left\{ \begin{array}{l} \{0\} \\ \{0, 1\} \end{array} \right\} \\
 &\iff s_i \in T_{x_0}
 \end{aligned}$$

Let $k_s := |K_s|$ and $Q \in [k]^{k_s}$.
 Then Q determines a unique order
 preserving map

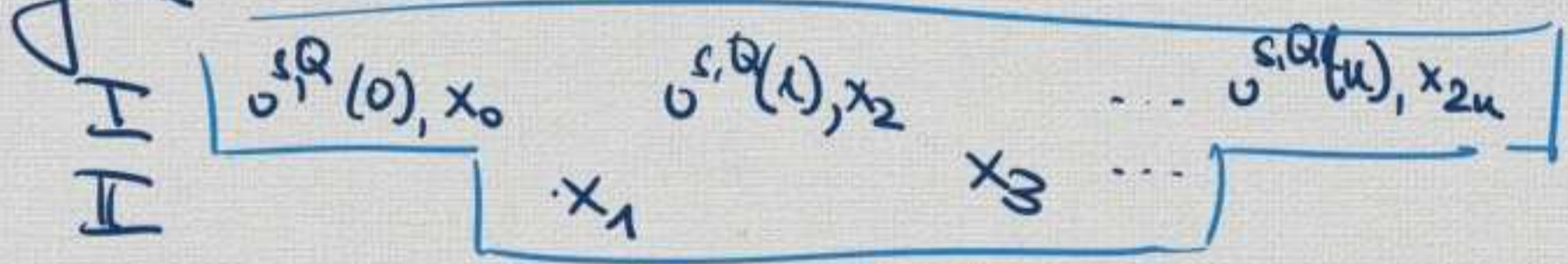
$$w_Q: (K_s, \prec_{K_s}) \longrightarrow (Q, \prec)$$

Define $v^{s,Q}(i) := w \upharpoonright K_{sti}$.

Then $v^{s,Q}$ is coherent with s . This
 allows me to translate a position in
 the original game



into a (legal) position in the auxiliary
 game



$s^* Q$
 Position n
 $M < \omega \times \omega < \omega$

Define a colouring $c_s : [k]^{ks} \rightarrow \omega$ $s \in \omega^{<\omega}$

by $c_s(Q) := \underline{\tau(s_*^Q)}$

$Q \in [k]^{ks}$

where τ is our w.s. for player II in $G_{\text{aux}}(\hat{\Gamma})$.

This is a family of countably many colourings, so by Rowbottom, we find $H \subseteq \kappa$, $|H| = \kappa$ s.t.

$c_s \upharpoonright [H]^{ks}$ is constant.

[i.e., $Q, Q' \in [H]^{ks}$, then $c_s(Q) = c_s(Q')$,

so $\tau(s_*^Q) = \tau(s_*^{Q'})$.

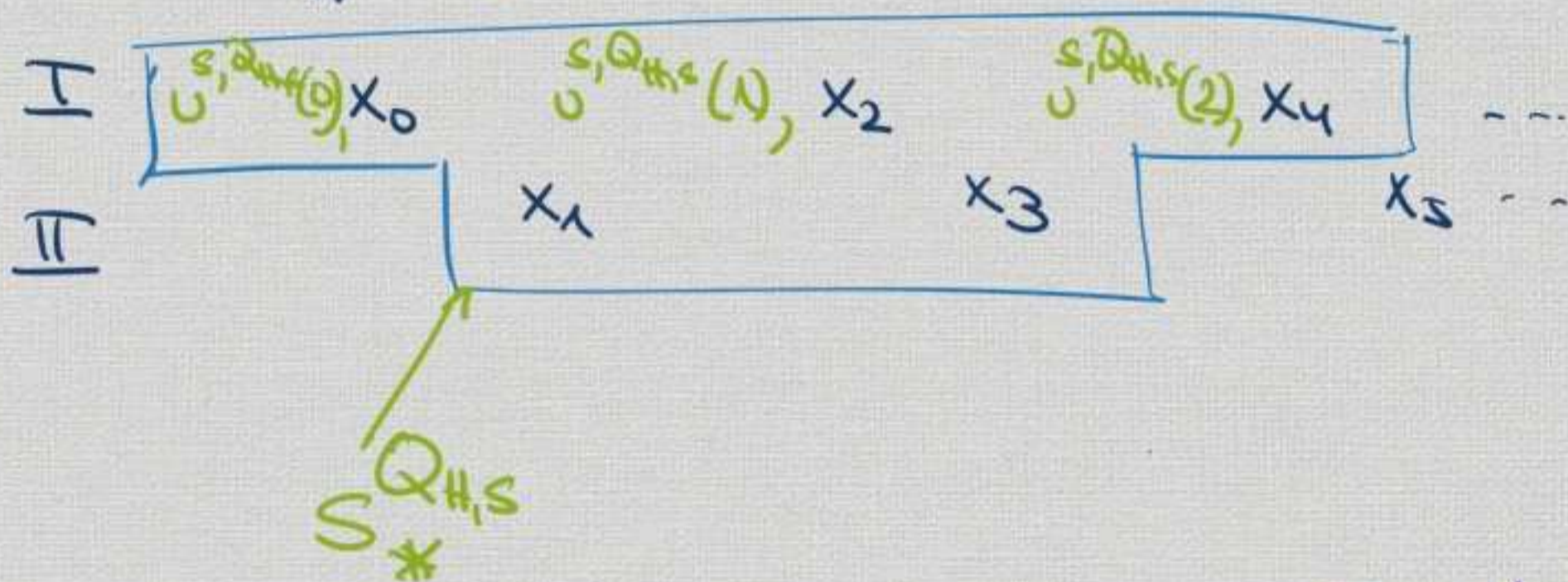
In words: if II fills in the gaps by any $Q \in [H]^{ks}$, then the answer of τ does not depend on the precise choice of Q .

So, let

$Q_{\#s} :=$ the first k_s many elements of $\#$.

Define strategy $\tau_{\#}$ in the game $G(A)$ for player $\#$:

$$\tau_{\#}(s) := \tau\left(s_{\#}^{Q_{\#s}}\right)$$



CLAIM: If τ was winning in $G_{\text{aux}}(\hat{T})$,
then $\tau_{\#}$ is winning in $G(A)$.