Lecture XVIII

INFINITE GAMES

Lent 2021
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$\kappa \longrightarrow (\kappa)^2_2$: κ is weakly compact

For all $c : [\kappa]^2 \longrightarrow 2$ there is H with $|H| = \kappa$ s.t. $c \upharpoonright [H]^2$ is constant.

FACTS: (1) Every weakly compact cardinal is inaccessible.

(2) Every measurable cardinal is weakly compact.

(1) is on Example Sheet #3 (29)

We're going to see a weak version of (1):

Proposition (ZFC). $\aleph_1$ is not weakly compact.

Pf. By AC, we have s.o.c. $C \subseteq WF$ with $|C| = \aleph_1$. Write

$$C = \{ x_\alpha ; \alpha < \omega_1 \}$$

with $\|x_\alpha\| = \alpha$.

Consider the lexicographic order on ω^ω:

$$x <_L y \iff x \restriction n = y \restriction n \text{ and } x(n) < y(n)$$

[n is unique]

If $(y_\alpha;\ \alpha<\gamma)$ is any increasing or decreasing sequence in the order $<_L$, then I can isolate the elements of the sequence by the basic open sets:

$$y_\alpha \restriction n = y_{\alpha+1} \restriction n$$

$$y_\alpha(n) < y_{\alpha+1}(n) \quad [>]$$

Take $s_\alpha := y_\alpha \restriction n+1$. Then $[s_\alpha] \ni y_\alpha$ but for no other element y_β, we have $y_\beta \in [s_\alpha]$.

Since there are only ctbly many b.o.s., we see that γ must be a ctble ordinal.

i.e., $x_\alpha <_L x_\beta \iff \alpha < \beta$

Define $c : [\aleph_1]^2 \longrightarrow 2$ by

$$c(\alpha,\beta) := \begin{cases} 1 & \text{if } <_L, < \text{ agree on } \alpha,\beta \\ 0 & \text{o/w} \end{cases}$$

if H has card. $\aleph_1$ and is c-homogeneous for colour 1, then it defines an $<_L$-increasing seq. in C; otherwise, if it's c-homogeneous for colour 0, then it defines a $<_L$-decreasing seq. in C. Both are contradictions!

q.e.d.

Now ② : Every measurable cardinal is weakly compact.

<u>Remark.</u> Example (30) gives a proof of ② without additional assumptions.

We're are going to see a slightly diff. proof that makes another assumption.

<u>Def.</u> If $X_\alpha \subseteq \kappa$ are subsets of κ for $\alpha < \kappa$, we call the set

$$\mathop{\triangle}_{\alpha<\kappa} X_\alpha := \{\gamma < \kappa ; \gamma \in \bigcap_{\alpha<\gamma} X_\alpha\}$$

the diagonal intersection.

Def. An ultrafilter U is called <u>normal</u> if it is closed under diagonal intersections.

Prop. If U is an ultrafilter on κ such that all elements of U have size κ and U is normal, then U is <u>κ-complete</u>.

Pf. Let X_α $(\alpha < \lambda)$ be in U.

Want to show: $\bigcap_{\alpha<\lambda} X_\alpha \in U$.

Define
$$Y_\alpha := \begin{cases} X_\alpha & \alpha < \lambda \\ \kappa & \alpha \geq \lambda. \end{cases}$$

Every $Y_\alpha \in U$, so $\underset{\alpha<\kappa}{\triangle} Y_\alpha \in U$ by normality
$$\underset{\alpha<\kappa}{\triangle} Y_\alpha = \{\gamma \,;\, \gamma \in \bigcap_{\alpha<\gamma} Y_\alpha\}.$$

Since $\lambda < \kappa$, we have $\lambda \notin U$, so $\kappa \setminus \lambda \in U$, so $Y := \left(\underset{\alpha<\lambda}{\triangle} Y_\alpha\right) \setminus \lambda \in U$.

Let $\eta \in Y$. Then $\eta \geq \lambda$.

$$\left.\eta \in \bigcap_{\alpha<\eta} Y_\alpha \subseteq \bigcap_{\alpha<\lambda} Y_\alpha = \bigcap_{\alpha<\lambda} X_\alpha\right\} \Rightarrow Y \subseteq \bigcap_{\alpha<\lambda} X_\alpha$$

$$\Downarrow$$

$$\bigcap_{\alpha<\lambda} X_\alpha \in U$$

Fact (ZFC).
If κ is measurable, then there is a normal ultrafilter on κ.

[Proof skipped.]

Theorem (ZFC). Measurable cardinals are weakly compact.

Pf. Let U be a normal nonprincipal κ-complete ultrafilter on κ. Let

$$c : [\kappa]^2 \longrightarrow 2$$

be any 2-colouring.

If $\alpha \in \kappa$, we write

$$c_\alpha(\beta) := \begin{cases} c(\alpha,\beta) & \alpha \neq \beta \\ 0 & \alpha = \beta \end{cases}$$

$$X_\alpha^0 := \{\beta \,;\, c_\alpha(\beta) = 0\}$$

$$X_\alpha^1 := \{\beta \,;\, c_\alpha(\beta) = 1\}$$

There is $i_\alpha \in \{0,1\}$ s.t. $X_\alpha^{i_\alpha} \in U$.

$I_0 := \{\alpha \,;\, i_\alpha = 0\}$ Either I_0 or $I_1 \in U$.

$I_1 := \{\alpha \,;\, i_\alpha = 1\}$ w.l.o.g. $I_0 \in U$.

Define

$$X_\alpha := \begin{cases} X_\alpha^0 & \text{if } \alpha \in I_0 \\ \kappa & \text{o/w} \end{cases}$$

By assumption, all $X_\alpha \in U$.

Thus $\underset{\alpha<\kappa}{\triangle} X_\alpha \in U$.

And $H := I_0 \cap \underset{\alpha<\kappa}{\triangle} X_\alpha \in \underline{U}$.

If I can show that H is c-homogeneous for colour 0, I am done [since $|H| = \kappa$].

Let $\underline{\alpha < \beta}$, $\alpha, \beta \in H$. $\longrightarrow$ $\alpha, \beta \in I_0$

$\longrightarrow X_\alpha = X_\alpha^0, X_\beta = X_\beta^0$

$\downarrow$

$$\beta \in \underset{\delta<\kappa}{\triangle} X_\delta = \{\gamma ; \gamma \in \bigcap_{\delta<\gamma} X_\delta\}$$

$$\longrightarrow \beta \in \bigcap_{\delta<\beta} X_\delta \subseteq X_\alpha = X_\alpha^0$$

$$\longrightarrow c_\alpha(\beta) = 0$$

$$\longrightarrow c(\alpha, \beta) = 0$$

q.e.d.

Tree Representations

We proved C closed $\Longleftrightarrow$ there is T tree on ω s.t. $C = [T]$ $\quad \underline{\mathrm{I}}$

A analytic $\Longleftrightarrow$ there is T tree on $\omega \times \omega$ s.t. $A = p[T]$ $\quad \underline{\mathrm{II}}$

We also saw that tree representation of type $\underline{\mathrm{I}}$ are important for determinacy proofs:

GALE-STEWART (even w/o AC)
If T is a tree on a wellordered set X, then $A \subseteq X^\omega$ with $A = [T]$ is determined.

__Q.__ Can we lift a determinacy argument for tree repr. of type $\underline{\mathrm{I}}$ to type $\underline{\mathrm{II}}$?

__Def.__ Let κ be a cardinal and T be a tree on $\kappa \times \omega$ [note that $\kappa \times \omega$ is wellordered]. We define
$$p[T] := \{ x \in \omega^\omega ; \exists y \in \kappa^\omega \ (y, x) \in [T] \}$$

Remark If $\kappa = \aleph_0$, then this is exactly the analytic sets [follows from our tree representation of type II].

Def. If κ is a cardinal and $A \subseteq \omega^\omega$, we say A is κ-Suslin if there is a tree T on $\kappa \times \omega$ s.t.

$$A = p[T].$$

By Remark, being $\aleph_0$-Suslin $\iff$ analytic.

Hope: Prove a Gale-Stewart-like Thm for κ-Suslin sets.

AUXILIARY GAMES

If A is κ-Suslin, say $A = p[T]$ for some T on $\kappa \times \omega$, we define the auxiliary game

$$G_{aux}(T) \text{ as follows:}$$

$\in \kappa$ $\quad$ $\in \kappa$

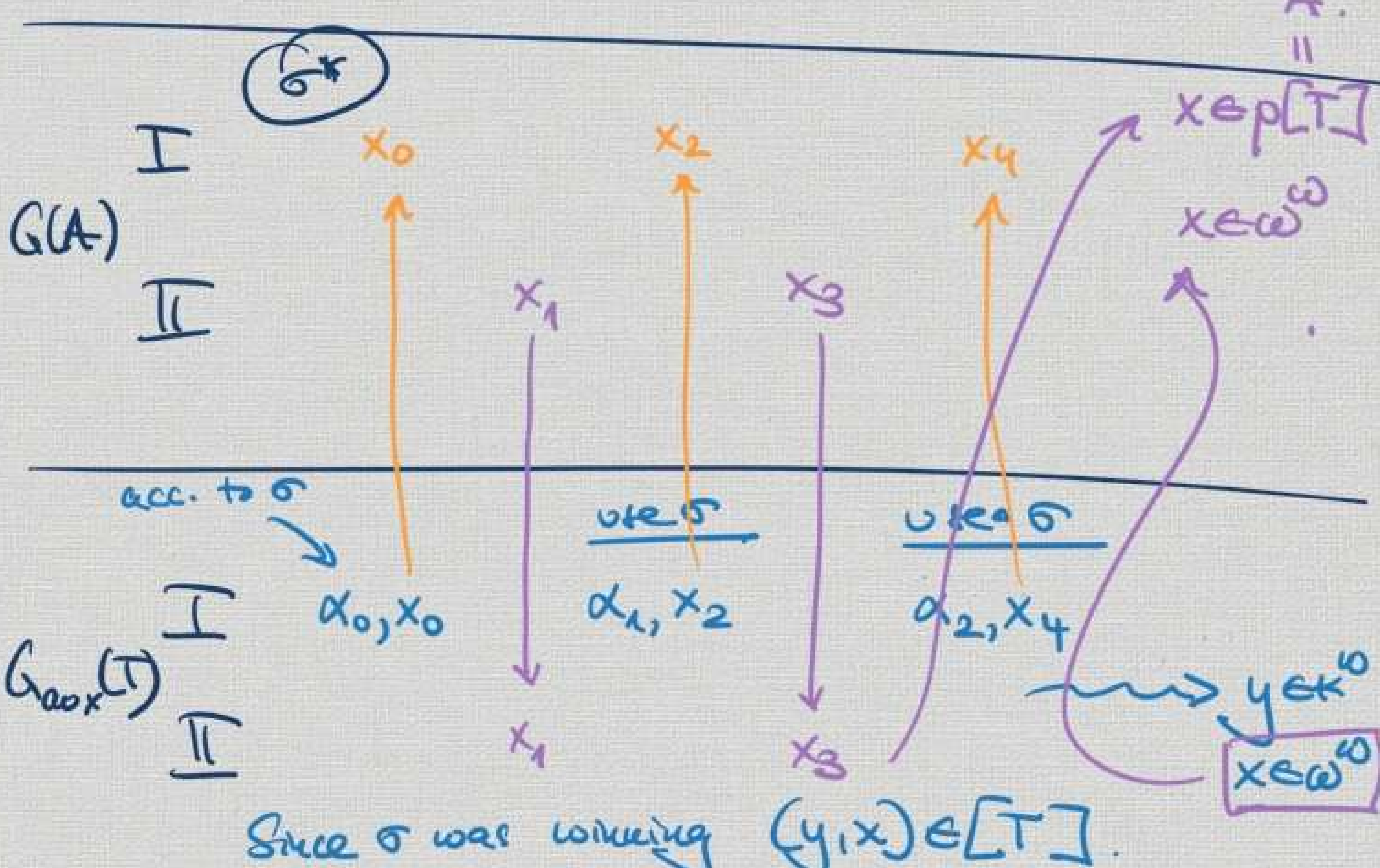
I	α_0, x_0	α_1, x_2	α_2, x_4	$\cdots$	
II		x_1	x_3	x_5	$\cdots$

$\in \kappa$

$y(i) = \alpha_i$
$x(i) = x_i$
$y \in \kappa^\omega$
$x \in \omega^\omega$

I wins if $(y, x) \in [T]$.

Relationship between $G(A)$ and $G_{aux}(T)$:



Suppose player I wins $G_{aux}(T)$ by σ, then the above diagram constructs a w.s. σ^* that is winning in $G(A)$.

What about the other direction?

If pl. II wins in $G_{aux}(T)$ ...?