\[ k \rightarrow (k^2)^2 \quad : \quad k \text{ is weakly compact} \]

For all \( c : (k^2)^2 \rightarrow 2 \) there is \( H \) with \( \text{lh}(H) = k \) s.t. \( c^{\text{lh}(H)^2} \) is constant.

**FACTS:**
1. Every weakly compact cardinal is inaccessible.
2. Every measurable cardinal is weakly compact.

(1) is on Example Sheet #3 (29)

We're going to see a weak version of (1):

**Proposition (29):** \( \omega_1 \) is not weakly compact.

**Proof:** By AC, we have s.o.c. \( C \subseteq WF \) with \( |C| = \omega_1 \). Write

\[
C = \prod_{\alpha < \omega_1} \alpha < \omega_3
\]

with \( \|x\alpha\| = \alpha \).

Consider the lexicographic order on \( \omega_1 \):
If \((y_a : a < x)\) is any increasing or decreasing sequence in the order \(\leq\), then I can isolate the elements of the sequence by the basic open sets:

\[
y_a \uparrow u = y_{a+1} \uparrow u \quad y_a(u) < y_{a+1}(u) \quad [\geq]
\]

Take \(s_a := y_a \uparrow u+1\). There \([s_a] \ni y_a\) but for no other element \(y_a\), we have \(y_a \in [s_a]\).

Since there are only countably many \(b.o.s.\), we see that \(y\) must be a club ordinal.

Define \(c : [\alpha, \beta] \rightarrow 2\) by:

\[
c(\alpha, \beta) := \begin{cases} 1 & \text{if } \langle \alpha, \beta \rangle \text{ agree on } \alpha \cap \beta \\ 0 & \text{otherwise} \end{cases}
\]
If \( H \) has cod. \( \Delta_1 \) and is \( c \)-homogeneous for colour 1, then it defines an \( \leq \)-increasing seq. in \( C \); otherwise, if it's \( c \)-homogeneous for colour 0, then it defines a \( \leq \)-decreasing seq. in \( C \). Both are contradictions.

\[ \text{q.e.d.} \]

Now (2): Every measurable codedical is \( \nabla \)-weakly compact.

**Remark.** Example (30) gives a proof of without additional assumptions.

We're going to see a slightly different proof that makes use of another.

**Def.** If \( X_\alpha \subseteq K \) are subsets of \( K \) for \( \alpha \in \kappa \), we call the set
\[ \Delta X_\alpha := \{ \overline{\{ \overline{\{ \overline{x \in X_\alpha} \}} \} : \alpha \in \kappa, x \in \bigcap_{\alpha < \kappa} X_\alpha \} \} \]
the diagonal intersection.
Def: An ultrafilter $U$ is called **normal** if it is closed under diagonal intersections.

Prop: If $U$ is an ultrafilter such that all elements of $U$ have size $\kappa$ and $U$ is normal, then $U$ is $\kappa$-complete.

Proof: Let $X_\alpha$ ($\alpha < \kappa$) be in $U$. We want to show: $\bigcap_{\alpha < \kappa} X_\alpha \in U$.

Define $Y_\alpha := \left\{ x : x \in X_\alpha \text{ and } \alpha \geq \lambda \right\}$. Every $Y_\alpha \in U$, so $\bigtriangleup Y_\alpha \in U$ by normality.

Since $\lambda < \kappa$, we have $\lambda \notin U$, so $\kappa \leq \lambda$.

Let $Y := \bigcap_{\alpha < \kappa} Y_\alpha$. Then $Y \geq \lambda$.

$Y \subset \bigcap_{\alpha < \kappa} Y_\alpha \subset \bigcap_{\alpha < \kappa} X_\alpha = \bigcup_{\alpha < \kappa} X_\alpha \in U$.
Fact (2FC).
If $k$ is measurable, then there is a normal ultrafilter on $k$.
[Proof skipped.]

Theorem (2FC). Measurable cardinals are weakly compact, nonprincipal.
Let $U$ be a normal $k$-complete ultrafilter on $k$. Let $c : [k]^2 \to 2$
be any 2-colouring.
If $x \in k$, we write
\[
c_x(\beta) := \begin{cases} c(x, \beta) & x \neq \beta \\ 0 & x = \beta \end{cases}
\]

Then $x_0 := \{ \beta : c_x(\beta) = 0 \}$
\[
x_1 := \{ \beta : c_x(\beta) = 1 \}
\]
There is $i_x \in 0, 1, 3$ s.t. $x_{i_x} \in U$.
$I_0 := \{ x \in k : i_x = 0 \}$ either $I_0 \cap U$ or $I_1 \in U$.
$I_1 := \{ x \in k : i_x = 1 \}$ w.l.o.g. $I_0 \cap U$. 
Define
\[ X_\alpha = \int x^\alpha \quad \text{if } \alpha \in I_0 \]
\[ \triangle X_\alpha = 0 \text{ otherwise} \]

By assumption, all \( X_\alpha \in U \).

Thus
\[ \Delta X_\alpha \in U \]

And \( H := I_0 \cap \bigcap_{\alpha < k} \Delta X_\alpha \in U \)

If I can show that \( H \) is \( c \)-homogeneous for colour 0, I am done [since \( |H| = k \)].

Let \( \alpha < \beta \), \( \alpha, \beta \in H \).

Then
\[ x_\alpha^0 = x_\alpha, \quad x_\beta^0 = x_\beta \]

\[ \beta \in \bigcap_{\delta \leq \beta} X_{\delta} \leq X_\alpha \]

\[ c_\alpha(\beta) = 0 \]

\[ c(\alpha, \beta) = 0 \quad \text{q.e.d.} \]
Tree Representations

we proved C closed \iff \text{there is } T \text{ tree on } \omega \\
\text{s.t. } C = [T]

A analytic \iff \text{there is } T \text{ tree on } \omega \times \omega \\
\text{s.t. } A = p[T]

We also saw that there representation of type I are important for
determinacy proofs:

GALE-STEWART (even w/o AC)

If \( T \) is a tree on a wellordered
set \( X \), where \( A \subseteq X \omega \) with
\( A = [T] \) is determined.

Q. Can we lift a determinacy requirement
for tree repr. of type I to

type II?

Def. Let \( \kappa \) be a cardinal and \( T \)
be a tree on \( \kappa \times \omega \) [note that
\( \kappa \times \omega \) is wellordered]. We define
\( p[T] := p' \times \omega \omega \) \iff \( \exists y \in \kappa \omega \) \( (y, x) \in [T] \).
Remark: If $k = \omega_0$, then this is exactly the analytic sets [follows from our tree representation of type II].

Def: If $k$ is a cardinal and $A \subseteq \omega$, we say $A$ is $k$-Souslin if there is a tree $T$ on $k \times \omega$ s.t. $A = p[T]$.

By Remark, being $\omega_0$-Souslin $\iff$ analytic.

Hope: Prove a Gale-Stewart-like theorem for $k$-Souslin sets.

**Auxiliary Games**

If $A$ is $k$-Souslin, say $A = p[T]$ for some $T$ on $k \times \omega$, we define the auxiliary game $G_{\omega \times (CT)}$ as follows:

I. $s_0, x_0, x_1, x_2, x_3, x_4, \ldots$  \quad I wins if $(y_1, x) \in [T]$

II. $y_0, x_1, x_2, x_3, x_4, \ldots$  \quad II wins if $(y_1, x) \not\in [T]$.
Relationship between $G(A)$ and $G_{aux}(T)$.

Since $\sigma$ was winning $(y_{1x_0})_e[T]$. Suppose player I wins $G_{aux}(T)$ by $\sigma$, then the above diagram constructs a w.s. $\sigma^*$ that is winning in $G(A)$. What about the other direction? If pl. II wins in $G_{aux}(T)$ ...?