Lecture XVII

1 March 2021

INFINITE GAMES
LENT 2021

Theorem 1 If there is a measurable cardinal, then $\mathrm{Det}(\underset{\sim}{\Pi}^1_1)$ holds.

Theorem 2 If ZF+AD holds, then $\aleph_1$ is a measurable cardinal.

Measurable cardinals.

Let κ be an uncountable cardinal.
We say that $F \subseteq \wp(\kappa)$ is a filter on κ if

(a) $\kappa \in F$, $\emptyset \notin F$

$\longrightarrow$ (b) $A, B \in F \implies A \cap B \in F$

(c) $A \in F$, $B \supseteq A \implies B \in F$.

Intuition Filter collect the "big" subsets of κ.

A filter F is called an ultrafilter if for every $A \subseteq \kappa$ either $A \in F$ or $\kappa \setminus A \in F$.

A filter F is called λ-complete if for every $\gamma < \lambda$ and every family $\{A_\alpha ; \alpha < \gamma\} \subseteq F$, we have $\bigcap_{\alpha<\gamma} A_\alpha \in F$.

<u>Remark</u> If $\lambda = \aleph_0$, then this means closure under finite intersections [that follows from (b) by induction].

If $\lambda = \aleph_1$, then it means closure under ctble intersections, often called <u>σ-completeness</u>.

If $\alpha \in \kappa$, then $U_\alpha := \{X \subseteq \kappa ; \alpha \in X\}$ is an ultrafilter which is λ-complete for all λ. We call these <u>principal ultrafilters</u> and those not of this form <u>non-principal ultrafilters</u>.

So, if U is a non-principal ultrafilter, then for all $\alpha \in \kappa$,

$$\{\alpha\} \notin U.$$

Therefore, if it is λ-complete and $A \subseteq \kappa$ with $|A| < \lambda$, then $A \notin U$.

[Apply De Morgan + λ-completeness to the family $\{\{\alpha\} ; \alpha \in A\}$.]

Fact ZFC proves that the set of ultrafilters on κ has cardinality 2^{2^κ} [of those exactly κ many are principal].

Definition A cardinal κ is called **measurable** if there is a κ-complete non-principal ultrafilter on κ.

(*) [In particular, for such a U, all elements of U have cardinality κ.]

Theorem (ZFC) Every measurable cardinal is inaccessible.

We shall see later that ZF+AD $\Longrightarrow \aleph_1$ is measurable: Clearly, $\aleph_1$ is not inaccessible, so this proof needs AC !!!

Proof. We need to show regular & strong limit. Fix U κ-complete non-pr. on κ.

1. **Regular** Suppose not, so there is some sequence $(\gamma_\alpha\,;\ \alpha<\lambda)$ with $\lambda<\kappa$ and $\gamma_\alpha<\kappa$ s.t. $\kappa = \bigcup_{\alpha<\lambda} \gamma_\alpha$.

$$\kappa = \bigcup_{\alpha<\lambda} \gamma_\alpha \qquad \begin{array}{l}\gamma_\alpha < \kappa \\ \lambda < \kappa.\end{array}$$

By choice of U, $\gamma_\alpha \notin U$.

[Since κ is a cardinal and $\gamma_\alpha < \kappa$ and (x) on last page.]

By κ-completeness,

$$\kappa = \bigcup_{\alpha<\lambda} \gamma_\alpha \notin U.$$

Part 1. of this proof works in ZF w/o choice.

In contradiction to filter property (a).

2. Strong limit. To show: if $\lambda < \kappa$, then

$$2^\lambda \ngeq \kappa.$$

We'll show that by assuming that there is an injection

$$f : \kappa \longrightarrow 2^\lambda$$

and derive a contradiction.

2^λ ← the unique cardinal in bijection with the set

w.l.o.g. $\underset{\text{injection}}{f : \kappa \longrightarrow S}$ $\quad S = \{g \, ; \, g : \lambda \to 2\}$

So: $f(\alpha) : \lambda \longrightarrow 2.$

$f: \kappa \longrightarrow S = \{g;\, g: \lambda \longrightarrow 2\}$

For fixed $\gamma < \lambda$, we consider

$$A_\gamma^0 = \{\alpha;\ f(\alpha)(\gamma) = 0\}$$
$$A_\gamma^1 = \{\alpha;\ f(\alpha)(\gamma) = 1\}$$

$A_\gamma^0 \cup A_\gamma^1 = \kappa$ Thus, exactly one

$A_\gamma^0 \cap A_\gamma^1 = \emptyset$ of the two is in U.

Let $i_\gamma \in \{0,1\}$ be such that

$$A_\gamma^{i_\gamma} \in U.$$

$$\{A_\gamma^{i_\gamma};\ \gamma \in \lambda\} \subseteq U$$

Thus by κ-completeness,

$$\bigcap_{\gamma<\lambda} A_\gamma^{i_\gamma} \in U.$$

I claim that $\left|\bigcap_{\gamma<\lambda} A_\gamma^{i_\gamma}\right| \leq 1$ in contradiction to non-principality of U.

Suppose $\alpha \in \bigcap_{\gamma<\lambda} A_\gamma^{i_\gamma}$.

Consider $f(\alpha)$. Then $f(\alpha)(\gamma) = i_\gamma$

[since $\alpha \in A_\gamma^{i_\gamma}$]

so there is only one function f such that for all $\alpha, \alpha' \in \bigcap_{\gamma<\lambda} A_\gamma^{i_\gamma}$,

$$f = f(\alpha) = f(\alpha').$$

Injectivity of F implies that

$$\left|\bigcap_{\gamma<\lambda} A_\gamma^{i_\gamma}\right| \leq 1.$$

q.e.d.

[<u>Remark</u>. The use of AC is in the set-up of the proof which requires that 2^λ is wellorderable.]

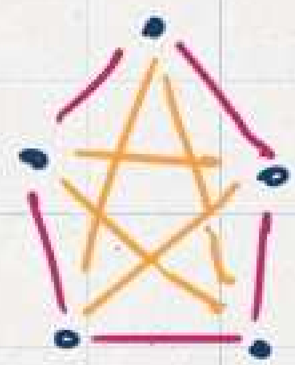
Erdős-Rado Arrow Notation

Def. Let κ, λ, μ be cardinals and $n \in \mathbb{N}$.

We write $[X]^n$ for the set of n-element subsets of X.

A function $c : [X]^n \longrightarrow \mu$ is called a *μ-colouring*.

If c is a μ-colouring, we call $H \subseteq X$ *c-homogeneous* or *c-monochromatic* if there some $\alpha \in \mu$ s.t. $\forall s \in [H]^n \;\; c(s) = \alpha$.



$c : [5]^2 \longrightarrow \{\bullet, \bullet\}$

We write

$$\kappa \longrightarrow (\lambda)^n_\mu$$

κ "arrows" $(\lambda)^n_\mu$.

for "every μ-colouring of $[\kappa]^n$ has a homogeneous subset H of cardinality λ".

Def. A cardinal κ is called weakly compact if

$$\kappa \longrightarrow (\kappa)^2_2 .$$

Fact (1) Every measurable cardinal is weakly compact. [Lecture XVIII]

(2) Every weakly compact cardinal is inaccessible. [Example Sheet #3].

Def We write $[X]^{<\omega}$ for the set of all finite subsets of X.

A function $c : [X]^{<\omega} \longrightarrow \mu$ is called a μ-colouring.

If $n \in \mathbb{N}$, $H \subseteq X$ we call H n-c-homogeneous if there is $\alpha \in \mu$ s.t. for all $s \in [H]^n$, $c(s) = \alpha$.

$\kappa \longrightarrow (\lambda)^{<\omega}_{\mu}$ if every μ-colouring has an n-c-homogeneous set of size λ for every $n \in \mathbb{N}$.

Theorem (Rowbottom's Theorem)

If κ is measurable and there is a countable set

$$\{ c_k \, ; \, k \in \mathbb{N} \}$$

with $c_k : [\kappa]^{<\omega} \longrightarrow \gamma$ for some $\gamma < \kappa$, then there is a set H of card. κ [actually in the ultrafilter U] s.t. H is n-c_k-homogeneous for all $n, k \in \mathbb{N}$ simultaneously.

[Example Sheet #3]