Sixteenth Lecture

INFINITE GAMES

LENT 2021
26 FEBRUARY 2021

Canonical model family

$x \in \omega^\omega$ $\underline{M}$, $\underline{M}_x$ s.t.

(a) $\forall x$ $\underline{M} \subseteq \underline{M}_x$

(b) $\forall x$ $x \in \underline{M}_x$

(c) $M \models GCH$

$\underset{\sim}{\Delta}^1_4$-wellordered

For each x, $<_x \subseteq \omega^\omega \times \omega^\omega$ s.t.

$\{(u,v) ; u, v \in \underline{M}_x \wedge u <_x v\}$

is a $\underset{\sim}{\Delta}^1_4$-wellordering.

Preservation of basic features in inner models:

$M \subseteq V$ $\quad$ M is transitive

$M, V \models ZFC$

We have (Example Sheet #3):

If $x, y, f \in M$

* $M \models f : x \longrightarrow y \iff V \models f : x \longrightarrow y$
* $M \models f$ is surjective $f : x \longrightarrow y$ $\iff$ $V \models f$ is surjective $f : x \longrightarrow y$
* $M \models x, y$ are ordinals & $f : x \longrightarrow y$ & f is cofinal $\iff$ $V \models x, y$ are ordinals & $f : x \longrightarrow y$ & f is cofinal.

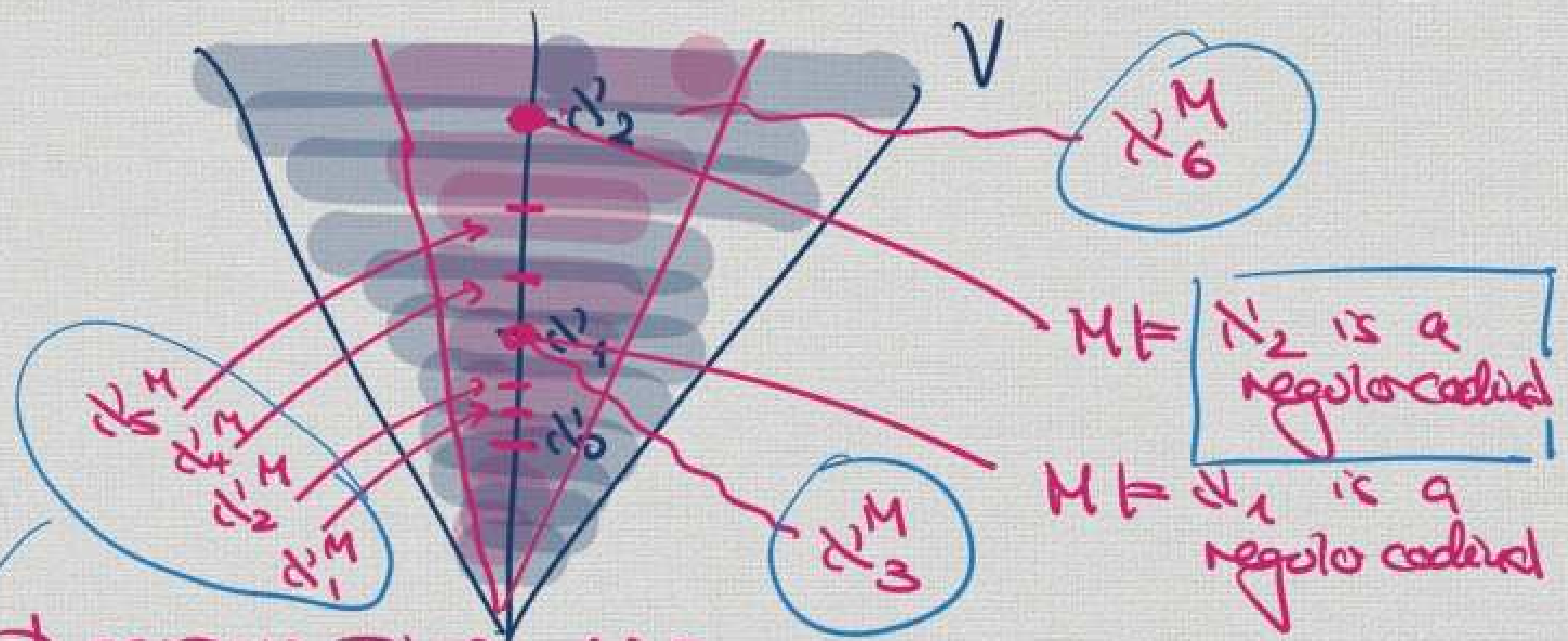
CONSEQUENCE

IN GENERAL $\not\Leftarrow$

If $V \models \kappa$ is a cardinal $\Longrightarrow M \models \kappa$ is a cardinal.

If $V \models \kappa$ is regular $\Longrightarrow M \models \kappa$ is regular.

$\not\Leftarrow$



POSSIBLY THERE ARE OTHER CARDINALS IN M.

If we write
"$\aleph_1$ is a cardinal"
we mean
"the first uncountable ordinal is a cardinal"

If we mean $\aleph_n$ to be interpreted by a formula in M, we write $\aleph_n^M$. If we mean the object interpreted in V, we write $\aleph_n = \aleph_n^V$.

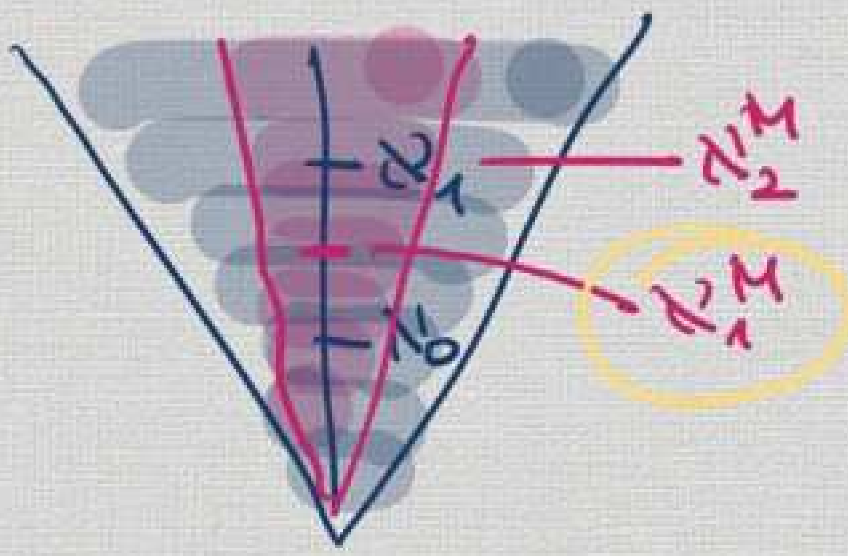
In general, $\aleph_n^M < \aleph_n^V$ is possible.

Another transfer between M & V:

$$M \models x \in WF \implies V \models x \in WF.$$

[EXAMPLE SHEET #3.]

$$M \models \alpha \text{ is countable}$$
$$\iff \exists x \ \|x\| = \alpha \ \& \ x \in WF \cap M$$
$$(*) \iff WF_\alpha \cap M \neq \emptyset$$



In this situation $\aleph_1^M < \aleph_1$, so in V,

$$WF_{\aleph_1^M} \neq \emptyset$$

But by $(*)$ $WF_{\aleph_1^M} \cap M = \emptyset$

since $M \models \aleph_1^M$ is not countable.

<u>Summary</u> $\aleph_1^M = \min\{\alpha \,;\, WF_\alpha \cap M = \emptyset\}$.

Let μ be a canonical model family with M, M_x.

$M \subseteq M_x$: $\aleph_1^M \leq \aleph_1^{M_x} \leq \aleph_1$

Theorem Suppose there is $\underset{\sim}{\Delta}^1_4$-wellordered canonical model family and $\mathrm{Det}(\underset{\sim}{\Pi}^1_4)$. Then there is an inner model of $\mathrm{ZFC} + \mathrm{IC}$.

there is an inaccessible cardinal

Proof. We fix models M, M_x from the canonical model family.

Def. $C \subseteq WF$ is called a weak set of unique codes if for all α $|WF_\alpha \cap C| \leq 1$.

We get from our general theorem about s.u.c.:

if C w.s.u.c., then C is uncountable $\iff$ C does not have the p.s.p.

By assumption, we know that

$\{(u,v) ; u,v \in M_x \ \& \ u <_x v\}$ is $\underset{\sim}{\Delta}^1_4$

The proof of "$\underset{\sim}{\Delta}^1_4$-wellord. $\Rightarrow$ s.u.c. that is $\underset{\sim}{\Pi}^1_n$" shows here that there is a w.s.u.c. C_x that is $\underset{\sim}{\Pi}^1_4$. $\underline{C_x \cap WF_\alpha = \emptyset \iff \alpha < \omega_1^{M_x}}$.

Clearly $|C_x| = |\aleph_1^{M_x}|$.

So C_x has the p.s.p. $\Longleftrightarrow$

is $\underset{\sim}{\Pi}^1_1$ $\quad \underline{\aleph_1^{M_x}} < \aleph_1$.

By Det($\underset{\sim}{\Pi}^1_1$), we get PSP($\underset{\sim}{\Pi}^1_1$) by asymmetric game. So for each x, C_x has the p.s.p., so

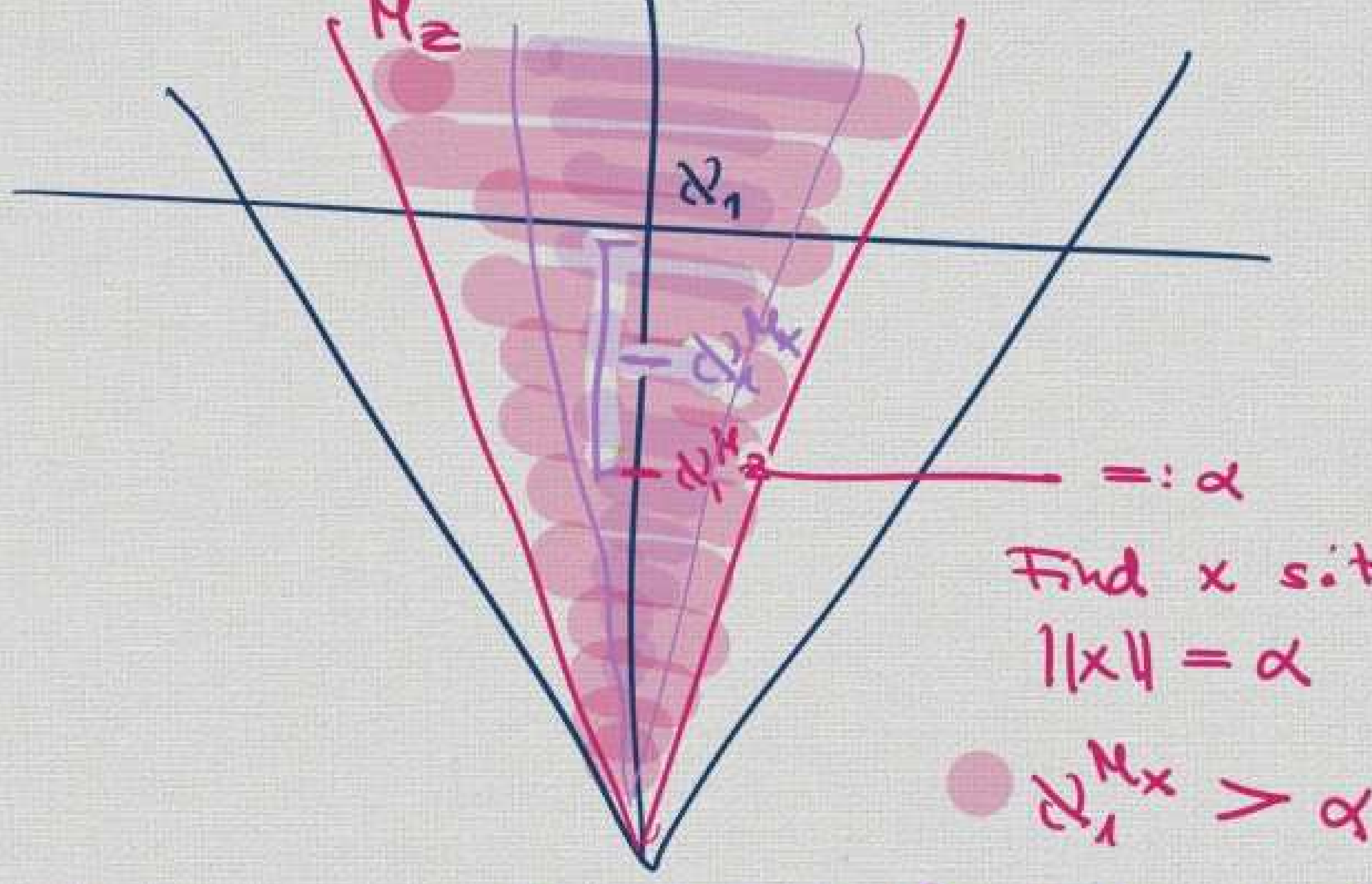
$$\forall x \quad \aleph_1^{M_x} < \aleph_1.$$

Also: if $\alpha < \aleph_1$, then $\underbrace{\ }_{\text{there is}}$ $x \in WF_\alpha$.

But then $x \in M_x$

$$\Longrightarrow WF_\alpha \cap M_x \neq \emptyset$$

$$\Longrightarrow \aleph_1^{M_x} > \alpha.$$



This argument shows that for each $\alpha < \aleph_1$ there is some x s.t. $\alpha < \aleph_1^{M_x} < \aleph_1$.

CLAIM $M \models \aleph_1^V$ is inaccessible.

Since $M \subseteq M_x$, we have $\aleph_1^M \leq \aleph_1^{M_x} < \aleph_1^V$.

Inaccessible means: regular strong limit.

Regular Being regular is preserved. ✓

Strong limit. Since $M \models GCH$, strong limit is eq. to limit. So I need to show that if $\kappa < \aleph_1$, λ is ordinal $M \models \lambda = \kappa^+$, then $\lambda < \aleph_1$.

① $\kappa < \lambda$ cardinals
② $\kappa < \aleph_1$
③ $M \models \kappa^+ = \lambda$
$\Bigg\} \Longrightarrow \lambda < \aleph_1$.

Let $x \in WF_\kappa$. [By ②, this exists in V.]

$x \in M_x \underset{\text{before}}{\Longrightarrow} \aleph_1^{M_x} > \kappa$.

Since $M \subseteq M_x$: $\aleph_1^{M_x}$ is a cardinal in M

& by ③ λ is the smallest cardinal $> \kappa$ in M.

$$\Longrightarrow \lambda \leq \aleph_1^{M_x} < \aleph_1.$$

This concludes the proof of the theorem.

q.e.d.

Next Move from inaccessible cardinals to measurable cardinals

Thm 1 If there is a measurable card., then $\mathrm{Det}(\Pi^1_1)$.

Thm 2 If ZF+AD ("all sets det."), $\aleph_1$ is a measurable card.