FIFTEENTH LECTURE

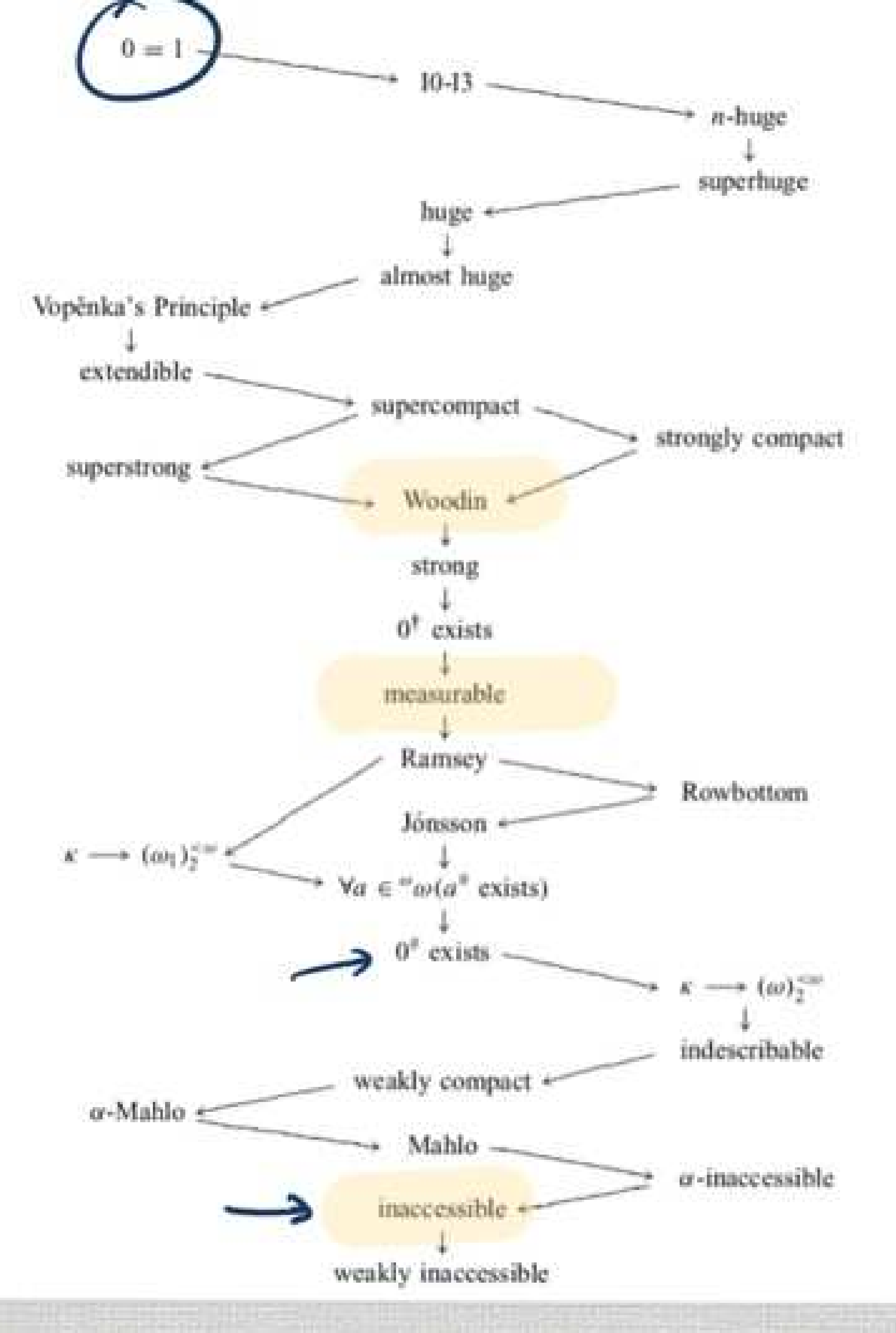
INFINITE GAMES
LENT 2021
Wednesday, 24 February 2021

GOAL: Relationship between
1. LARGE CARDINALS
2. DETERMINACY
3. DEFINABLE WELL-ORDERINGS OF ω^ω.

LARGE CARDINALS

Chart of Cardinals from: Kanamori, The Higher Infinite

The arrows indicates direct implications or relative consistency implications, often both.



IMPRECISE DEFINITION

Let Φ be a property of cardinals. We write
ΦC for $\exists x\, \Phi(x)$.

Φ is going to be the LARGE CARDINAL PROPERTY

ΦC is going to be the LARGE CARDINAL AXIOM

Φ is a LCP if

1. $\Phi(\kappa)$ implies that κ is "large" in some sense.
2. $\Phi C \vdash \mathrm{Cons}(ZFC)$

② $ZFC + \Phi C \vdash Cons(ZFC)$

Comments.

(i) Gödel's Incompleteness Theorem says if $ZFC \vdash \Phi C$ and ②

then ZFC is inconsistent.

So, under reasonable assumptions, it means

$$ZFC \nvdash \Phi C.$$

(ii) Gödel's Completeness Theorem tells us that

$$Cons(ZFC)$$

is equivalent to

$$\exists M \text{ "} M \models ZFC \text{"}.$$

If Φ, Ψ are two L.C.P., we can say

$\underline{\Phi C < \Psi C}$ if $ZFC + \Psi C \vdash Cons(ZFC + \Phi C)$

ΦC & ΨC are equiconsistent

if $ZFC + Cons(ZFC + \Phi C) \longleftrightarrow ZFC + Cons(ZFC + \Psi C)$

Def. A cardinal is called inaccessible if it is regular and a strong limit.
[A cardinal κ is a strong limit if
for all $\lambda < \kappa$, $2^\lambda < \kappa$.]

We write I for the property of being inaccessible, as before

$IC :\iff \exists x\, I(x)$

"there is an inaccessible cardinal".

Remark. κ is a limit $\iff$ $\forall \lambda < \kappa\, (\lambda^+ < \kappa)$.

Remark. Under GCH $[\forall \lambda\ \lambda^+ = 2^\lambda]$ we have:
κ is limit $\iff$ κ is strong limit.

Check properties (1) & (2).

(1) Clearly, if $I(\kappa)$, then
$$\aleph_1, \aleph_2, \aleph_3 < \kappa$$
$$\aleph_\omega, \aleph_{\omega_1}, \aleph_{\omega_2} \ldots < \kappa$$
if $\lambda = \aleph_\lambda$ [Aleph fixed pt] and λ is least with that property, then $cf(\lambda) = \aleph_0 < \kappa$, so $\lambda < \kappa$.

(2) Need to show $ZFC + IC \vdash Cons(ZFC)$.

<u>Theorem</u> If κ is inaccessible, then $V_\kappa \models ZFC$

VON NEUMANN HIERARCHY

- $V_0 := \emptyset$
- $V_{\alpha+1} := \mathcal{P}(V_\alpha)$
- $V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$.

(*) In L&ST 2019/20, Example Sheet #4, Example 9: If $\lambda > \omega$ limit ordinal, then

$V_\lambda \models$ all axioms of ZFC minus Replacement.

$\underbrace{\text{all axioms of ZFC minus Replacement}}_{\text{Zermelo Set Theory}}$

So: we need to show that if κ is inacc., then $V_\kappa \models$ Replacement.

<u>Lemma 1</u> If κ is inaccessible, then $\forall \alpha < \kappa$

$$|V_\alpha| < \kappa.$$

<u>Pf.</u> Follows directly from the two conditions in the definition by transfinite induction.

q.e.d.

Lemma 2 Let κ be inaccessible. Then TFAE:

(i) $x \in V_\kappa$

(ii) $x \subseteq V_\kappa$ & $\underline{|x| < \kappa}$.

Proof (i) $\Rightarrow$ (ii).

$x \in V_\kappa \Longrightarrow x \subseteq V_\kappa$ [transitivity of V_κ]

$x \in V_\kappa = \bigcup_{\alpha<\kappa} V_\alpha \Longrightarrow$ ex. $\alpha < \kappa$ $x \in V_\alpha$

$\Longrightarrow x \subseteq V_\alpha$

$\Longrightarrow |x| \leq |V_\alpha| \underset{L1}{<} \kappa$.

(ii) $\Rightarrow$ (i).

Suppose $x \subseteq V_\kappa$. For every $y \in x$, define $\alpha_y := \rho(y)$ $\quad \underline{\underline{y \in V_{\alpha_y+1} \setminus V_{\alpha_y}}}$.

$A := \{ \alpha_y^{+1} ; y \in x \}$

Clearly $|A| \leq |x| < \kappa$..

By regularity of κ, $\bigcup A =: \alpha < \kappa$.

But then for all $y \in x$, $y \in V_\alpha$. $\Rightarrow$ $x \subseteq V_\alpha$ $\Downarrow$ $x \in V_{\alpha+1} \subseteq V_\kappa$

Proof of Thm

We are going to show something even stronger:

Take any function $F : V_\kappa \to V_\kappa$ and show that if $x \in V_\kappa$, then $F[x] \in V_\kappa$.

If $x \in V_\kappa$, by L2, we know that $|x| < \kappa$, but then

$$\left.\begin{array}{l} |F[x]| \leq |x| < \kappa. \\ \text{Clearly, } F[x] \subseteq V_\kappa. \end{array}\right\} \overset{L2}{\Longrightarrow} F[x] \in V_\kappa.$$

q.e.d.

Next goal:

If there is a family of models with nice wellorders of ω^ω and $PSP(\underset{\sim}{\Pi}^1_1)$ holds, then there is a model of $ZFC + IC$.

1. M is an inner model if it is a transitive class containing all ordinals and a model of ZFC.

1.* Suppose $(V, \in) \models \mathrm{ZFC}$ is a set model. Then $M \subseteq V$ is called an inner model if it contains all of the ordinals of V and is transitive in V and $(M, \in) \models \mathrm{ZFC}$.

[EXAMPLE SHEET #3]

2. A formula φ defines an inner model if

$$x \in M \longleftrightarrow \varphi(x).$$

3. A formula μ is called a canonical model family if the following classes are inner models:

$$M := \{w \,;\, \mu(\emptyset, w)\} \quad \text{ROOT}$$

For each $x \in \omega^\omega$: $M_x := \{w \,;\, \mu(x, w)\}$

with the properties: (a) $\forall x \;\; M \subseteq M_x$

(b) $\forall x \;\; x \in M_x$

(c) $M \models \mathrm{GCH}$.

4. If μ is a canonical model family, we say that μ is

<u>$\underset{\sim}{\Delta}^1_n$ - wellordered</u>

if for each x, there is a wellordering $<_x$ of $\omega^\omega \cap M_x$ s.t.

$$\{(u,v);\ u,v \in M_x \wedge u <_x v\}$$

is $\underset{\sim}{\Delta}^1_n$.