Lecture XIX

INFINITE GAMES
LENT 2021
5 March 2021

(1) A is <u>κ-Suslin</u> if there is a tree T on $\omega \times \kappa$ such that $A = p[T]$

(2) A is $\aleph_0$-Suslin iff A is analytic ($\underset{\sim}{\Sigma}^1_1$)

(3) If A is κ-Suslin (say, $A = p[T]$ for some tree T on $\omega \times \kappa$, define <u>DETERMINED</u> auxiliary game $G_{aux}(T)$

(*) s.t. if I has w.s. in $G_{aux}(T)$, then I has a w.s. in $G(A)$.

(4) Find κ with added combinatorial structure s.t. if II has w.s. in $G_{aux}(T)$, then II has w.s. in $G(A)$.

$\boxed{\mathrm{Det}(\underset{\sim}{\Pi}^1_1)}$

<u>Rough idea</u> Fix κ measurable.

(a) Show that every $\underset{\sim}{\Pi}^1_1$ set is κ-Suslin.

(b) Use measurability to prove the translation for player II [from (3)]

(c) Use determinacy of $G_{aux}(T)$ to get determinacy of $G(A)$.

Is it possible that this is an accurate description?

<u>NO</u>! If every set that is κ-Suslin (for κ meas.) was determined, then this proof would give much more: AD. This can't be!

Suppose that

if A is κ-Suslin & κ is measurable $\Longrightarrow A$ is determined. (*)

Observe: if $\lambda < \kappa$ and A is λ-Suslin, then A is κ-Suslin.

Also: every set A is $2^{\aleph_0}$-Suslin [Example Sheet #3]

If κ is measurable, then $2^{\aleph_0} < \kappa$, so every set is κ-Suslin.

So if (*) is true in this abstract form, every set is determined.

<u>We conclude</u>: It can't be just the fact that A is κ-Suslin, but we must require some properties of the tree T that witnesses $A = p[T]$.

Use our structure theory for $\underset{\sim}{\Pi}^1_1$ sets in order to define a tree on $\kappa \times \omega$ for $\underset{\sim}{\Pi}^1_1$ sets.

<u>ROADMAP</u>. If $A \in \underset{\sim}{\Pi}^1_1$, there is a tree T on $\omega \times \omega$ s.t.

$$x \in A \longleftrightarrow T_x \text{ is wellfounded}$$
$$\longleftrightarrow (T_x, \supsetneq) \text{ is wellfounded}$$
$$\longleftrightarrow \text{there is an order preserving map from } (T_x, \supsetneq) \text{ into } (\omega_1, <)$$

If we code T_x by some bijection $\omega^{<\omega} \longleftrightarrow \omega$ then the o.p. map is essentially an element of ω_1^ω.

<u>FIND</u> tree $\hat{T} \subseteq (\omega_1 \times \omega)^{<\omega}$ s.t. if $(y, x) \in [\hat{T}]$, then y is an order-pres. map from T_x into ω_1 [up to coding]

Then: $x \in A \longleftrightarrow \exists y \; (y, x) \in [\hat{T}]$.

INFORMAL DEFINITION OF $\hat{T}$: $(v, \underline{s}) \in \hat{T}$ where $v \in \omega_1^{<\omega}$, $s \in \omega^{<\omega}$

$\Longleftrightarrow$

v is consistent with being an initial segment of an o.p. map from T_x into ω_1. [where x is any ext. of s].

THEOREM (Shoenfield).

Every $\underset{\sim}{\Pi}^1_1$ set is κ-Suslin (for every $\kappa \geq \omega_1$).

Proof. Fix A $\underset{\sim}{\Pi}^1_1$. Fix T s.t.

$$x \in A \longleftrightarrow T_x \text{ is wellfounded.}$$

Fix $i \longmapsto s_i$ bijection between ω and $\omega^{<\omega}$ with property that if $s_i \subseteq s_j$, then $i \leq j$.

[This implies that $\mathrm{lh}(s_i) \leq i$.]

$$T_x = \{t \in \omega^{<\omega} ; (t, x \upharpoonright \mathrm{lh}(t)) \in T\}$$

$$T_s := \{t \in \omega^{<\omega} ; (t, s \upharpoonright \mathrm{lh}(t)) \in T\}$$

$T_x = \bigcup_{n \in \mathbb{N}} T_{x \upharpoonright n}$

T_s is a tree that has finitely many levels, if $t \in T_s$, then $\mathrm{lh}(t) \leq \mathrm{lh}(s)$.

But it could be an <u>infinite</u> set.

$$K_x := \{i \in \omega ; s_i \in T_x\}$$

$$K_s := \{i \in \omega ; i \leq \mathrm{lh}(s) \text{ and } s_i \in T_s\}$$

Then K_s is <u>finite</u>!

$$T_x = \{s_i ; i \in K_x\}$$

In general, $T_s \supsetneq \{s_i ; i \in K_s\}$, but

$$K_x = \bigcup_{n \in \mathbb{N}} K_{x \upharpoonright n}.$$

Observe that $M := \{ p \,;\, p$ is a partial function from ω to κ with finite domain$\}$ has cardinality κ, so it's sufficient to provide a tree T on $M \times \omega$ with the desired properties (rather than on $\kappa \times \omega$).

We can define a linear order on $\kappa^{<\omega}$ that extends the order $\supsetneq$, known as the KLEENE-BROUWER order:

$$s <_{KB} t \;:\Longleftrightarrow\; t \supsetneq s \text{ or}$$

there is $i \in \mathrm{dom}(s) \cap \mathrm{dom}(t)$ s.t. $s(i) \neq t(i)$; let i be least such AND $s(i) < t(i)$.

$\boxed{\text{It's strict extension if it applies and lexicographic otherwise.}}$

Intuitively, consider $x_s : \omega \longrightarrow \kappa \cup \{\infty\}$ defined by

$$x_s(i) := \begin{cases} s(i) & \text{if } i \in \mathrm{dom}(s) \\ \infty & \text{otherwise} \end{cases}$$

and then order these lexicographically.

Clearly, $<_{KB}$ extends $\supsetneq$ and is total.

Fact If T is a tree then

$(T, \supsetneq)$ is wellfounded $\iff$

$(T, <_{KB})$ is a wellorder.

[Example Sheet #3].

Now let's define $\hat{T} \subseteq (M \times \omega)^{<\omega}$.

If $v \in M^{<\omega}$ and $s \in \omega^{<\omega}$, we say that v is coherent with s if

① $lh(v) \leq lh(s)$

② $i < lh(v) \implies dom(v(i)) = K_{s \upharpoonright i}$.

③ $i < lh(v) \implies v(i) : (K_{s \upharpoonright i}, <_{KB}) \to \kappa$ is o.p.

④ $i \leq j \implies v(i) \subseteq v(j)$

$$\hat{T} := \{(s, v) ; v \text{ is coherent with } s\}$$

the Shoenfield tree

CLAIM : $A = p[\hat{T}]$.

1. $\mathrm{lh}(u) \leq \mathrm{lh}(s)$
2. $i < \mathrm{lh}(u) \Rightarrow \mathrm{dom}(u(i)) = K_s \upharpoonright i$
3. $i < \mathrm{lh}(u) \Rightarrow u(i)$ is o.p.
4. $i \leq j \Rightarrow u(i) \subseteq u(j)$.

CLAIM $\quad A = p[\hat{T}]$

"$\subseteq$". Suppose $x \in A$. By ass. T_x is wellfounded, so

$$(T_x, <_{KB}) \text{ is a wellorder}$$

so there is an o.p. function

$$g : (T_x, <_{KB}) \longrightarrow (\kappa, <)$$

Translate into K_x by

$$h : (K_x, <_x) \longrightarrow (\kappa, <)$$

where $i <_x j :\Longleftrightarrow s_i <_{KB} s_j$

This is order preserving.

Define $u(i) := h \upharpoonright K_x \upharpoonright i$.

Then clearly ②, ③, and ④ are true.

So for every i $(u \upharpoonright i, x \upharpoonright i) \in \hat{T}$.

So $(u,x) \in [\hat{T}]$. So $x \in p[\hat{T}]$.