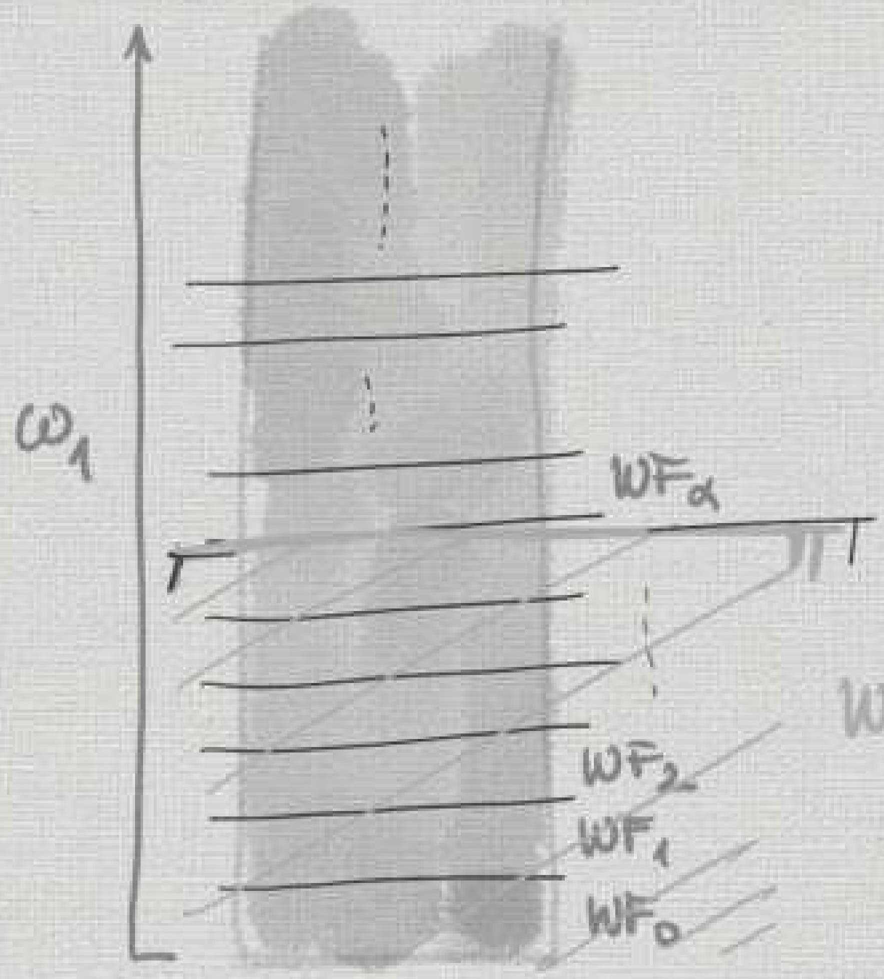
Lecture XIV

INFINITE GAMES
LENT 2021

22 February 2021



STRATIFICATION OF THE SET WF

$$WF = \bigcup_{\alpha<\omega_1} WF_\alpha = \bigcup_{\alpha<\omega_1} WF_{<\alpha}$$

- For each α $WF_\alpha, WF_{<\alpha}, WF_{\le\alpha}$ are $\underset{\sim}{\Delta}^1_1$ (Borel).
- WF is $\underset{\sim}{\Pi}^1_1$-complete.
- So every $\underset{\sim}{\Pi}^1_1$ set A is an ω_1-union of Borel sets.

Proposition If $A \in \underset{\sim}{\Pi}^1_1$, then $|A|$ is either $\leq \aleph_0$, or $\aleph_1$, or $2^{\aleph_0}$.

["Weak Continuum Hypothesis for $\underset{\sim}{\Pi}^1_1$ etc"]

Proof. By our analysis, we know

$$A = \bigcup_{\alpha < \omega_1} A_\alpha \quad \text{where } A_\alpha \text{ is Borel.}$$

By Borel determinacy & the characterisation of PSP in terms of games:

A_α has the p.s.p. f. each α.

So for each α, A_α is ctble or there is a T_α perfect s.t. $[T_\alpha] \subseteq A_\alpha$.

Case 1. There is some α s.t. the second case holds: $[T_\alpha] \subseteq A_\alpha \subseteq A$,

so A contains a perfect subset: $|A| = 2^{\aleph_0}$.

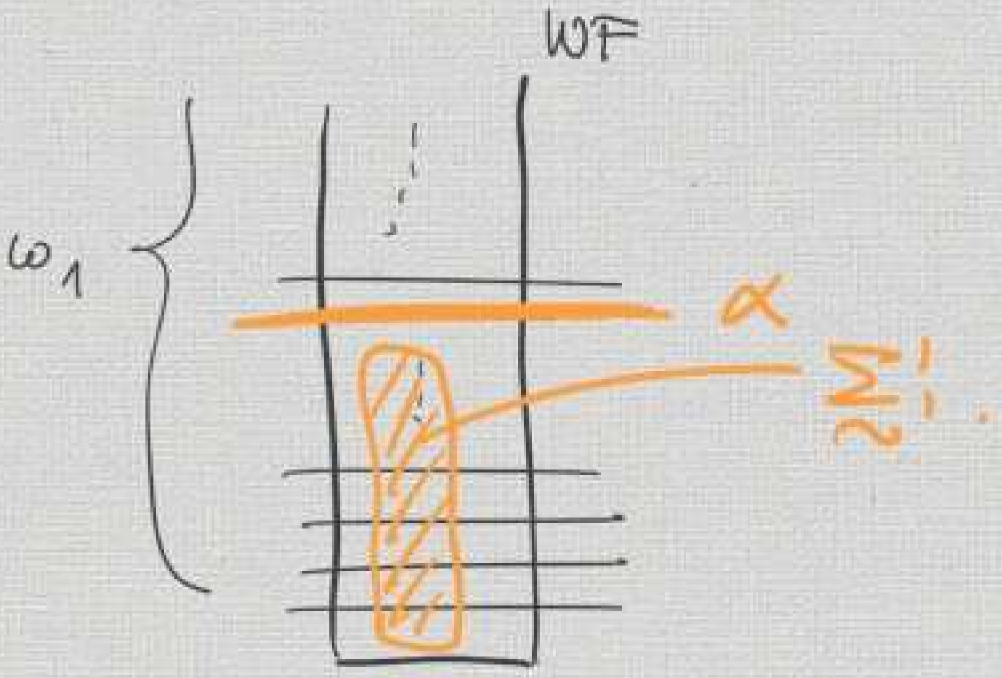
Case 2 All A_α are countable

$$|A| \leq \aleph_1 \cdot \aleph_0 = \aleph_1. \quad \text{q.e.d.}$$

Theorem BOUNDEDNESS LEMMA

If $A \subseteq WF$ s.t. A is Σ^1_1
then there is an $\alpha < \omega_1$ s.t.

$$A \subseteq WF_{<\alpha}.$$



Proof. Let us write
$OP(y, x, z)$ for
"y is an order-preserv. map from $(\mathrm{fld}(x), R_x)$ to $(\mathrm{fld}(z), R_z)$"

In the proof last lecture we saw that this is a closed set.

[This allows us to express $\|x\| = \|z\|$, then this is just $\underbrace{\exists y \exists y' \underbrace{OP(y,x,z) \wedge OP(y',z,x)}_{\Pi^0_1}}_{\Sigma^1_1}$]

We prove bddness by contradiction:

(*) Assume $A \in \boldsymbol{\Sigma}^1_1$, $A \subseteq WF$ unbounded and show that $WF \in \boldsymbol{\Sigma}^1_1$.

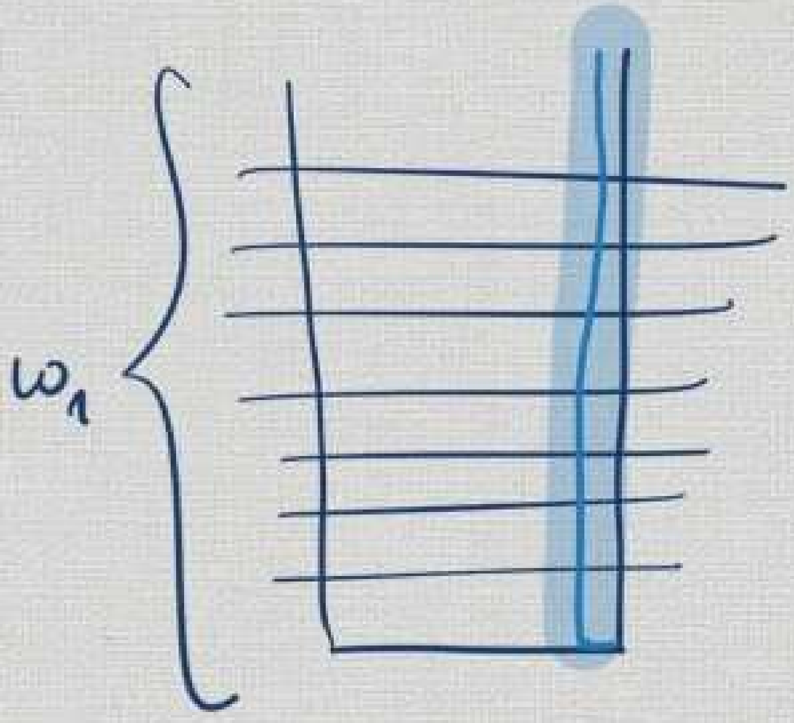
Under the assumption (*), we have

$$x \in WF \iff \exists a \, \big(a \in A \wedge \exists y \; OP(y, x, a) \big).$$

$$\underbrace{\exists a \, \exists y \, \underbrace{\big(\underbrace{a \in A}_{\boldsymbol{\Sigma}^1_1} \wedge \underbrace{OP(y, x, a)}_{\boldsymbol{\Pi}^0_1} \big)}_{\boldsymbol{\Sigma}^1_1}}_{\boldsymbol{\Sigma}^1_1}.$$

This contradicts the $\boldsymbol{\Pi}^1_1$-completeness of WF.

q.e.d.



<u>Def.</u> $C \subseteq WF$ is called a <u>set of unique codes</u> if for all $\alpha \in \omega_1$,

$$|C \cap WF_\alpha| = 1.$$

Clearly, if C is a s.u.c., then

$$|C| = \aleph_1.$$

Also clearly, AC implies the existence of s.u.c.:

$\{WF_\alpha \,;\, \alpha \in \omega_1\}$ is a family of non-empty sets.

A choice fn for this family gives a s.u.c. as range.

So, the fragment of choice needed is

$$AC_{\omega_1}(\omega^\omega).$$

<u>Theorem</u> If C is a s.u.c., it cannot have the p.s.p.

<u>Pf</u>. Since C is unctble, if it has the p.s.p., it must contain some $[T] \subseteq C$ with T perfect.

closed

$[T] \subseteq C \subseteq WF$
so $[T]$ is a $\underset{\sim}{\Sigma}^1_1$ subset of WF

By Boundedness, we find $\alpha < \omega_1$ s.t. $[T] \subseteq \underbrace{WF_{<\alpha} \cap C}$.

$|WF_{<\alpha} \cap C| = |\alpha| \leq \aleph_0$.

This contradicts the fact that
$$|[T]| = 2^{\aleph_0}.$$
q.e.d.

Theorem If there is a $\underset{\sim}{\Delta}^1_4$ wellorder of ω^ω, then there is a $\underset{\sim}{\Pi}^1_4$ set without the perfect set property.

Proof. Produce a set of unique codes. In each WF_α, there is a $<$-least element if $<$ is the $\underset{\sim}{\Delta}^1_2$-wellorder.

$$C := \{x \,;\, \exists \alpha < \omega_1 \text{ } x \text{ is the } <\text{-least element of } WF_\alpha\}$$

is a s.u.c. and thus doesn't have the p.s.p.

We need to analyse the definition of C a bit better.

What does it mean to be in C?

$x \in WF$ and if $z \in WF$ and $\underbrace{\|x\| = \|z\|}_{\exists y \exists y' \, OP(y, x, z) \wedge OP(y', z, x)}$, then $x \leq z$ in the $\underset{\sim}{\Delta}^1_4$-wellorder.

$x \in C \iff$

$$\underbrace{\underbrace{x \in WF}_{\boldsymbol{\Pi}^1_1} \wedge \underbrace{\forall z \Big(\underbrace{\underbrace{\exists y \exists y' \underbrace{OP(y,x,z) \wedge OP(y',z,x)}_{\boldsymbol{\Pi}^0_1}}_{\boldsymbol{\Sigma}^1_1} \rightarrow \underbrace{x \leq z}_{\boldsymbol{\Delta}^1_n}}_{\substack{\text{union of } \boldsymbol{\Pi}^1_1 \text{ and } \boldsymbol{\Delta}^1_n \\ \text{if } n \geq 2, \text{ this is } \boldsymbol{\Delta}^1_n \\ \text{if } n = 1, \text{ this is } \boldsymbol{\Pi}^1_1}} \Big)}_{\boldsymbol{\Pi}^1_n}}_{\boldsymbol{\Pi}^1_n}$$

So C is a s.o.c. which is in $\boldsymbol{\Pi}^1_n$.

q.e.d.

CLOSE CONNECTION BETWEEN GAMES, WELLORDERS, AND LARGE CARDINALS

LARGE CARDINALS		DETERMINACY		DEFINABLE WELLORDERS
ω many Woodin cardinal + measurable above	$\longleftrightarrow$	Det(Definable)	$\longleftrightarrow$	no definable wellorders
ω many Woodin cardinals	$\longleftrightarrow$	Det(Proj)	$\longleftrightarrow$	no projective wellorders
n Woodin cardinals + a measurable above them	$\longleftrightarrow$	$\mathrm{Det}(\boldsymbol{\Pi}^1_{n+1})$	$\longleftrightarrow$	no $\boldsymbol{\Delta}^1_{n+2}$ wellorders
$\kappa<\lambda$, κ is Woodin, λ is measurable	$\longleftrightarrow$	$\mathrm{Det}(\boldsymbol{\Pi}^1_2)$	$\longleftrightarrow$	no $\boldsymbol{\Delta}^1_3$ wellorders
measurable cardinal	"$\longleftrightarrow$" $\rightleftarrows$	$\mathrm{Det}(\boldsymbol{\Pi}^1_1)$	$\longleftrightarrow$	no $\boldsymbol{\Delta}^1_2$ wellorders
ZFC		$\mathrm{Det}(\boldsymbol{\Delta}^1_1)$		—

These results are Martin-Steel theorem of projective determinacy (1985).