Infinite Games – Lecture XIII

Lent 2021 • 19 February 2021

TREES

$T \subseteq \omega^{<\omega}$

illfounded: $[T] \neq \emptyset$

wellfounded: $[T] = \emptyset$

bij. $\mathbb{N} \leftrightarrow \omega^{<\omega}$
$\{s_i \,;\, i \in \omega\}$

$F_T := \{i \in \omega \,;\, s_i \in T\}$
$R_T := \{(i,j) \,;\, s_i \supsetneq s_j\}$

RELATIONS ON $\mathbb{N}$

$F \subseteq \omega$
$R \subseteq F \times F$

$\underline{A = (F,R)}$ $\left.\begin{matrix}\text{ill-}\\ \text{well-}\end{matrix}\right\}$ founded

ORDINALS

If (F,R) is wellfdd., there is $\alpha < \omega_1$ s.t.

$\mathrm{rk}: F \longrightarrow \alpha$

is an order preserving map.

$\mathrm{rk}(i) = \sup\{\mathrm{rk}(j)+1 \,;\, iRj\}$

$x_A(\ulcorner i,j \urcorner) := \begin{cases} 1 & \text{if } iRj \\ 0 & \text{o/w} \end{cases}$

$\mathrm{fld}(x) := \{i \,;\, x(\ulcorner i,i \urcorner) \neq 0\}$
$R_x := \{(i,j) \,;\, x(\ulcorner i,j \urcorner) \neq 0\}$

ELEMENTS OF BAIRE SPACE

$x \in \omega^\omega$

Tree Representation for $\mathbf{\Pi}^1_1$ sets: A is $\mathbf{\Pi}^1_1$ iff

$\exists T\, \forall x\, (x \in A \longleftrightarrow [T_x] = \emptyset)$

Def. $WF \subseteq \omega^\omega$

$$WF := \{x \in \omega^\omega;\ (\mathrm{fld}(x), R_x) \text{ is wellfounded}\}$$

Then the rank function

$$\mathrm{rk} : \mathrm{fld}(x) \longrightarrow \alpha$$

gives an order preserving map from $\mathrm{fld}(x)$ into $\alpha =: \mathrm{ht}(\mathrm{fld}(x), R_x)$

HEIGHT

If $x \in WF$, $\|x\| := \mathrm{ht}(\mathrm{fld}(x), R_x)$.

This operation $\|\cdot\| : WF \longrightarrow \omega_1$ is a surjection.

[Let $\alpha < \omega_1$. There is some injection

$$f : \alpha \longrightarrow \mathbb{N}.$$

Define $F := \mathrm{ran}(f)$

$$f(\beta)\, R\, f(\gamma) \;:\Longleftrightarrow\; \beta \leq \gamma$$

Then by construction $(F, R) \cong (\alpha, \leq)$!

So if $A := (F, R)$ and $x := x_A$,

then $\|x_A\| = \alpha$.]

Define:

$$WF_\alpha := \{ x \in WF ; \|x\| = \alpha \}$$

$$WF_{<\alpha} = \bigcup_{\beta < \alpha} WF_\beta$$

$$WF_{\leq\alpha} = \bigcup_{\beta \leq \alpha} WF_\beta$$

Thus WF can be thought of as STRATIFIED in ω_1 many levels.

Theorem WF is $\underset{\sim}{\Pi}^1_1$.

Proof. If $A = (F, R)$ is a relation on $\mathbb{N}$, we can define

$y \in \omega^\omega$ is an A-descending seq. :

$$\forall i \quad y(i+1) \, R \, y(i) \wedge y(i+1) \neq y(i)$$

$x \in WF \iff \forall y \; y$ is not an $(\mathrm{fld}(x), R_x)$-descending seq.

$x \notin WF \iff \exists y \; y$ is a $(\mathrm{fld}(x, R_x))$-descending seq.

$$x \notin WF \iff \exists y \; \boxed{\begin{array}{c} \forall i \;\; x(\ulcorner y(i+1), y(i) \urcorner) \neq 0 \\ \text{and } \; y(i+1) \neq y(i) \end{array}}$$

$$C := \{ (y,x) ; \; \forall i \;\; x(\ulcorner y(i+1), y(i) \urcorner) \neq 0 \text{ and } y(i+1) \neq y(i) \}$$

C is closed in $\omega^\omega \times \omega^\omega$

By definition $\omega^\omega \setminus WF$ is $\underset{\sim}{\Sigma}^1_1$.

Thus WF is $\underset{\sim}{\Pi}^1_1$. q.e.d.

general proof technique extracted from this:

if C is $\underset{\sim}{\Pi}^1_n$ / $\underset{\sim}{\Sigma}^1_n$ $(\omega^\omega \times \omega^\omega)$ and

$$x \in A \iff \exists y \; / \; \forall y \; (y,x) \in C$$

then A is $\underset{\sim}{\Sigma}^1_{n+1}$ / $\underset{\sim}{\Pi}^1_{n+1}$

EXAMPLE of complexity calculation.

$$x \in A \iff \underbrace{\exists y \; \underbrace{\forall z \; \underbrace{\ldots\ldots}_{}}_{\underset{\sim}{\Sigma}^1_2 \;\;\; \underset{\sim}{\Pi}^1_3}}_{\underset{\sim}{\Sigma}^1_4}$$

Now check the complexity of the sets

WF_α, $WF_{<\alpha}$, $WF_{\leq\alpha}$. $WF_{<\alpha} = WF \cap N_\alpha$

$x \in WF_{<\alpha} \iff x \in WF$ and there is no orderpreserving map from α into $(\mathrm{fld}(x), R_x)$.

$N_\alpha := \{ x ;$ there is no o.p. map from α into $(\mathrm{fld}(x), R_x) \}$

[Fix some $a \in \omega^\omega$ s.t. $(\mathrm{fld}(a), R_a) \cong (\alpha, \leq)$ [we saw that this exists]]

$= \{ x ;$ there is no o.p. map from $(\mathrm{fld}(a), R_a)$ into $(\mathrm{fld}(x), R_x) \}$

$= \{ x ; \forall y$ it is not the case that $[\forall i \; a(\ulcorner i,i \urcorner) \neq 0 \Rightarrow x(\ulcorner y(i), y(i) \urcorner) \neq 0$ and $\forall i,j \; a(\ulcorner i,j \urcorner) \neq 0 \Longrightarrow x(\ulcorner y(i), y(j) \urcorner) \neq 0] \}$

So N_α is $\underset{\sim}{\Pi}^1_1$. and thus $WF_{<\alpha}$ is $\underset{\sim}{\Pi}^1_1$.

Similarly $WF_{\leq\alpha}$, WF_α.

$WF_{\leq\alpha}$ is also $\underset{\sim}{\Sigma}^1_1$.

$$WF_{\leq\alpha} = \{ x ; (\mathrm{fld}(x), R_x) \text{ o.p. maps into } (\mathrm{fld}(\alpha), R_\alpha) \}$$

$$x \in WF_{\leq\alpha} \iff \underbrace{\exists y \; \underbrace{\forall i,j \;\; x(\ulcorner i,j \urcorner) \neq 0 \longrightarrow \alpha(\ulcorner y(i), y(j) \urcorner) \neq 0}_{\text{CLOSED}}}_{\underset{\sim}{\Sigma}^1_1}.$$

SUMMARY

For every $\alpha < \omega_1$,

WF_α, $WF_{<\alpha}$, $WF_{\leq\alpha}$ are $\underset{\sim}{\Delta}^1_1$. Borel

EXAMPLE SHEET #2:

$\underset{\sim}{\Delta}^1_1$ = Borel.

Corollary WF can be written as a union of ω_1 many Borel sets.

$$WF = \bigcup_{\alpha<\omega_1} WF_\alpha$$

Def. Let Γ be a boldface pointclass.
A set $A \subseteq \omega^\omega$ is called Γ-hard
if for all $B \in \Gamma(\omega^\omega)$ there is
a cts function $f : \omega^\omega \longrightarrow \omega^\omega$
s.t. $B = f^{-1}[A]$
A is called Γ-complete if it's
Γ-hard and $A \in \Gamma(\omega^\omega)$.

Theorem WF is $\underset{\sim}{\Pi}^1_1$-complete.

Proof. Tree representation theorem:
if B is $\underset{\sim}{\Pi}^1_1$, then there is a tree
T s.t.

$$\forall x \cdot \underbrace{x \in B \Longleftrightarrow T_x \text{ is wellfounded}}$$

$$x \longmapsto c_{T_x} \in \omega^\omega \text{ s.t.}$$

$$c_{T_x}(\ulcorner i, j \urcorner) := \begin{cases} 1 & \text{if } s_i, s_j \in T_x \\ & \& \ s_i \supseteq s_j \\ 0 & \text{o/w} \end{cases}$$

Consider $x \longmapsto c_{T_x}$ and check that it is continuous.

$$x \longmapsto c_{T_x}$$

$$c_{T_x}(\ulcorner i,j \urcorner) := \begin{cases} 1 & s_i, s_j \in T_x \\ & s_i \supseteq s_j \\ 0 & \text{o/w} \end{cases}$$

Given i,j how much information about x do I need to determine whether

$$c_{T_x}(\ulcorner i,j \urcorner) = 0 \text{ or } 1 \text{ ?}$$

whether $s_i \supseteq s_j$ does not depend on x at all!

What does $s_i \in T_x$ mean?

$$(s_i, x \upharpoonright \mathrm{lh}(s_i)) \in T.$$

If I know $x \upharpoonright \max(\mathrm{lh}(s_i), \mathrm{lh}(s_j))$, then I can calculate $c_{T_x}(\ulcorner i,j \urcorner)$.

So $x \longmapsto c_{T_x}$ is continuous.

$$x \in B \longleftrightarrow [T_x] = \emptyset$$
$$\longleftrightarrow T_x \text{ is wellfdd}$$
$$\longleftrightarrow c_{T_x} \in WF$$

So B is the cts preimage of WF.

q.e.d.

<u>Corollary</u> WF is not $\underset{\sim}{\Sigma}^1_1$.

<u>Proof.</u> We know that $\underset{\sim}{\Pi}^1_1 \neq \underset{\sim}{\Sigma}^1_1$. However, if $B \in \underset{\sim}{\Pi}^1_1$ arbitrary by completeness of WF, if WF is $\underset{\sim}{\Sigma}^1_1$, B is $\underset{\sim}{\Sigma}^1_1$. Contradiction!

<u>Corollary</u> Every $\underset{\sim}{\Pi}^1_1$ set is an ω_1-union of Borel sets.

<u>Proof.</u> $B \in \underset{\sim}{\Pi}^1_1$, find f s.t. $B = f^{-1}[WF]$.

$$B = f^{-1}\Big[\bigcup_{\alpha<\omega_1} WF_\alpha\Big]$$
$$= \bigcup_{\alpha<\omega_1} \underbrace{f^{-1}[WF_\alpha]}_{\text{Borel.}}$$