Infinite Games Lecture XII

Lent 2021 17 February 2021

Theorem If Γ is a boldface pointclass, then $\mathrm{Det}(\Gamma) \Longrightarrow \mathrm{PSP}(\Gamma)$.

$A \subseteq 2^\omega$

$$\begin{array}{llllll} \mathrm{I} & s_0 & & s_1 & & \cdots \\ \mathrm{II} & & b_0 & & b_1 & \end{array} \qquad z = (s_0, b_0, s_1, b_1, \cdots)$$

$G^*(A)$ $\quad z^* := s_0 b_0 s_1 b_1 \ldots \in 2^\omega$

Player I wins if $z^* \in A$.

Claim 0 If $A \in \Gamma$, then there is $A^* \in \Gamma$ s.t. $G^*(A)$ & $G(A^*)$ are equivalent.

Claim 1 If I has a w.s. in $G^*(A)$, then A contains a perfect subset.

Claim 2 If II has a w.s. in $G^*(A)$, then A is ctble.

$p = (s_0, b_0, \ldots, s_n, b_n)$; τ strategy, $s \in 2^{<\omega}$

$\longrightarrow p s \tau := (s_0, b_0, \ldots, s_n, b_n, s, \tau(\cdots))$

$x \in 2^\omega$; p is **τ-decisive for x** if $p^* \subseteq x$, but for all $s \in 2^{<\omega}$ $p s \tau^* \not\subseteq x$.

Subclaim 2a If τ is winning for II, then for each $x \in A$ there is p that is τ-decisive for x.

Subclaim 2b Every p is τ-decisive for at most one $x \in 2^\omega$.

Conclusion of Subclaims 2a & 2b:

COUNTABLE

$$f : A \longrightarrow \mathrm{Pos}$$

$x \longmapsto$ the unique p s.t. p is τ-decisive for x

So this implies Claim 2.

p is *τ-decisive for x* if $p^* \subseteq x$, but for all $s \in 2^{<\omega}$, $ps\tau^* \not\subseteq x$.

Proof of Subclaim 2b

Let p be τ-decisive for x and show that every $x(k)$ is determined uniquely by p and τ.

By def. $p^* \subseteq x$

if $\ell := \text{lh}(p^*)$ & $k < \ell$, then

$x(k) = p^*(k)$, so determined by p.

Consider now

$x(\ell + n)$ where $n \in \mathbb{N}$.

We determine it recursively:

$x(\ell + 0) = x(\ell)$ if $s_0 := \emptyset$

$ps_0\tau^* \not\subseteq x$ (*)

$\text{lh}(\underset{\ell}{p}\,\underset{0}{s_0}\,\underset{1}{\tau^*}) = \ell + 1.$

So $ps_0\tau^*(\ell) \neq x(\ell)$ by (*)

Thus $x(\ell) = 1 - ps_0\tau^*(\ell)$

[since we're on Cantor space]

This determines $x(\ell)$ by just p, τ.

Now assume we know

$$x(\ell+0), \dots, x(\ell+n-1)$$

and determine $x(\ell+n)$.

Let $s_n := (x(\ell+0), \dots, x(\ell+n-1))$

$\quad lh(s_n) = n.$

Consider $p s_n \tau$. By decisiveness, we have $p s_n \tau^* \notin X$.

$\quad \ell \quad n \quad 1$

So by choice of s_n:

$$p s_n \tau^*(\ell+n) \neq x(\ell+n)$$

$$x(\ell+n) = 1 - p s_n \tau^*(\ell+n).$$

So, once more, $x(\ell+n)$ is determined just by p & τ.

q.e.d.
(Subclaim 2b)

$\Longrightarrow$ Thm.

<u>Corollary</u>. $ZFC \vdash PSP(\text{Borel})$.
(Hausdorff's Thm)

[Our proof is modulo Borel-Det.]

<u>And</u>: $\quad Det(\underset{\sim}{\Pi}^1_1) \Longrightarrow PSP(\underset{\sim}{\Pi}^1_1)$

$\quad Det(\underset{\sim}{\Pi}^1_n) \Longrightarrow PSP(\underset{\sim}{\Pi}^1_n)$

This yields necessary conditions for axioms of determinacy in the projective hierarchy:

e.g., if $\mathrm{Det}(\underset{\sim}{\Pi}^1_2)$, then we can't have $\underset{\sim}{\Pi}^1_2$ sets violating the CH.

The conditions are non-trivial.

<u>Theorem</u> (Gödel–Addison).

[without proof]

There is a model of $\mathrm{ZFC} + \neg\mathrm{PSP}(\underset{\sim}{\Pi}^1_1)$

<u>Remark</u>. This is "Gödel's Constructible Universe", usually denoted by $\mathbf{L}$. The reason for this is that $\mathbf{L}$ has a Δ^1_2 wellorder of ω^ω.

What does that even mean?

If $\leq$ is a wellorder of ω^ω, then it is a binary relation on ω^ω, so

$$\leq \; \subseteq \omega^\omega \times \omega^\omega$$

Therefore, it is perfectly reasonable to ask whether $\leq \in \underset{\sim}{\Delta}^1_2((\omega^\omega)^2)$

Theorem [our next goal]

If there is a $\underset{\sim}{\Delta}^1_4$ wellorder of ω^ω, then there is a set in $\underset{\sim}{\Pi}^1_4$ without the perfect set property.

Remark This is not optimal, as the Gödel–Addison theorem shows.

Proving this theorem will require:

① a structural analysis of $\underset{\sim}{\Pi}^1_1$

② a relation between $\underset{\sim}{\Pi}^1_1$ and the ordinal ω_1.

STRUCTURE THEORY OF CO-ANALYTIC SETS.

Tree representation theorem for closed set:

(*) $A \in \underset{\sim}{\Pi}^0_1 \iff$ there is a tree T s.t. $A = [T]$.

The pointclass $\underset{\sim}{\Sigma}^1_1$ was defined in terms of projections & closed sets.

$$A \in \underset{\sim}{\Sigma}^1_1 \iff \exists C \in \underset{\sim}{\Pi}^0_1 \text{ s.t. } A = pC.$$

$$\overset{(*)}{\iff} \exists T \text{ tree s.t. } A = p[T].$$

If T is a tree on $\omega\times\omega$ and $x\in\omega^\omega$, we can define

$$T_x := \{ s \,;\, (s, x\restriction \mathrm{lh}(s)) \in T \}$$

A is $\mathbf{\Sigma}^1_1 \iff \exists T$ tree s.t. $\forall x$ $(x\in A \iff [T_x] \neq \emptyset.)$

<u>Def.</u> A tree T is called <u>illfounded</u> if $[T]\neq\emptyset$ and <u>wellfounded</u> if $[T]=\emptyset$.

With some axiom of choice, this is equivalent to $(T,\supseteq)$ being ill/wellfounded.

Tree Representation of analytic and co-analytic sets.

$A\in\mathbf{\Sigma}^1_1 \iff \exists T$ tree s.t. $\forall x$ $(x\in A \iff T_x$ is illfounded$)$

$A\in\mathbf{\Pi}^1_1 \iff \exists T$ tree s.t. $\forall x$ $(x\in A \iff T_x$ is wellfounded$)$

Coding trees on ω or $\omega\times\omega$ as elements of Baire space.

Pick your favourite bijection $\omega \longrightarrow \omega^{<\omega}$ and write $\{s_i ; i \in \omega\} = \omega^{<\omega}$

Define R by $iRj :\Longleftrightarrow s_i \supseteq s_j$

$$F_T := \{i ; s_i \in T\}$$

Then $(T, \supseteq) \cong (F_T, R)$

In particular: T is wellfdd $\Longleftrightarrow$ (F_T, R) is wellfounded

If I have a wellfdd relation R on F, I can recursively define a rank function

$$i \in F \qquad rk(i) := \sup\{rk(j)+1 ; jRi\}$$

By the recursion theorem, if R is wellfdd, rk is a function assigning an ordinal to each elt of F:

$\{rk(i) ; i \in F\}$ is a countable ordinal

natural number

We now identify our relation on subsets of ω with elements of Baire space:

$$\ulcorner n,m \urcorner : \omega \times \omega \longrightarrow \omega$$

your favourite bijection

$x \in \omega^\omega$

$$\mathrm{fld}(x) := \{ i \,;\, x(\ulcorner i,i \urcorner) \neq 0 \}$$

$$R_x := \{ (i,j) \,;\, x(\ulcorner i,j \urcorner) \neq 0 \}$$

$(\mathrm{fld}(x), R_x)$ is a reflexive relation.

If (A,R) is any such structure, i.e.,

$A \subseteq \omega$, $R \subseteq A \times A$ reflexive.

Then
$$x_A(\ulcorner i,j \urcorner) := \begin{cases} 1 & \text{if } i,j \in A \\ & iRj \\ 0 & \text{o/w} \end{cases}$$

Then $\mathrm{fld}(x_A) = A$

$$iRj \iff i R_{x_A} j .$$