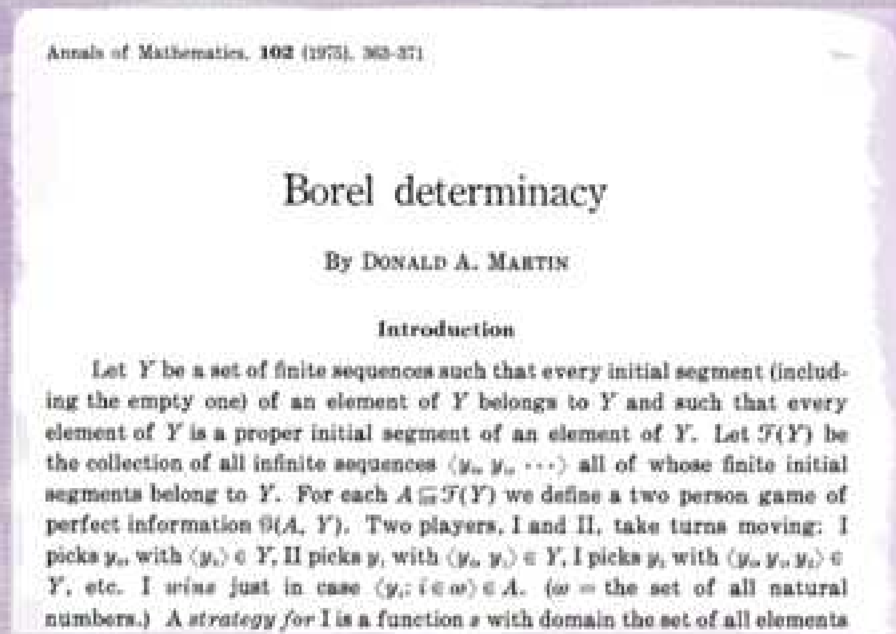
INFINITE GAMES, Lecture X

12 February 2021

Donald A. Martin (1973)



Annals of Mathematics, **102** (1975), 363-371

Borel determinacy

By DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initial segment (including the empty one) of an element of Y belongs to Y and such that every element of Y is a proper initial segment of an element of Y. Let $\mathcal{F}(Y)$ be the collection of all infinite sequences $\langle y_0, y_1, \cdots\rangle$ all of whose finite initial segments belong to Y. For each $A \subseteq \mathcal{F}(Y)$ we define a two person game of perfect information $\mathcal{G}(A, Y)$. Two players, I and II, take turns moving: I picks y_0, with $\langle y_0\rangle \in Y$, II picks y_1 with $\langle y_0, y_1\rangle \in Y$, I picks y_2 with $\langle y_0, y_1, y_2\rangle \in Y$, etc. I *wins* just in case $\langle y_i : i \in \omega\rangle \in A$. ($\omega$ = the set of all natural numbers.) A *strategy for* I is a function s with domain the set of all elements

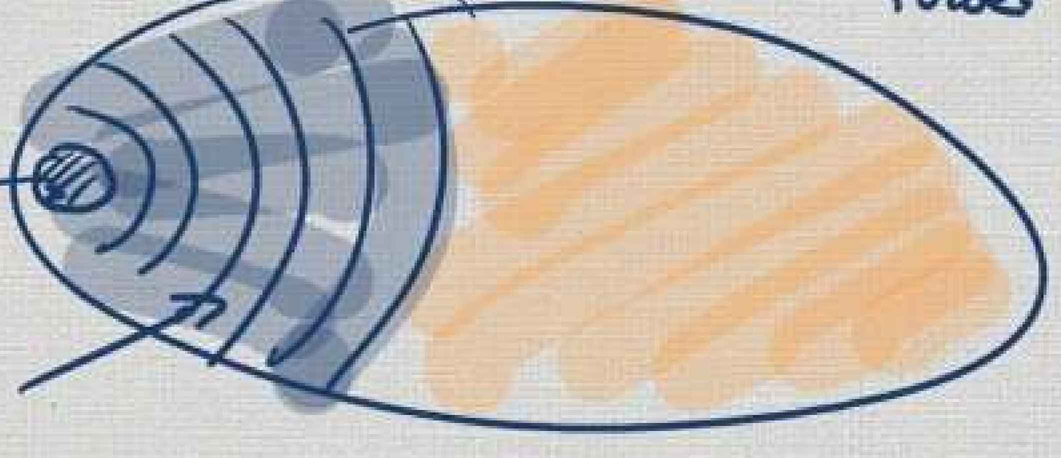
<u>Theorem</u> (ZFC). Det(Borel).

You could think:

Every reasonable set is Borel, so a proof that every Borel has some property is IN PRACTICE equivalent to proving that all sets have this property.

Size of this is $2^{2^{\aleph_0}}$

Power set of ω^ω



Σ^0_1

$\mathcal{B}$ Borel

What is the size of $\mathcal{B}$?

Clearly $|\mathcal{L}| \leq 2^{2^{\aleph_0}}$.

<u>Theorem (ZFC)</u>. $|\mathcal{L}| = 2^{\aleph_0} < 2^{2^{\aleph_0}}$.

<u>Proof</u>. Since $\{x\} \in \mathcal{L}$ for every $x \in \omega^\omega$

$x \longmapsto \{x\}$ is an inj. from ω^ω into $\mathcal{L}$.

So $2^{\aleph_0} \leq |\mathcal{L}|$.

<u>Upper bound</u> Proof by induction:

We prove that $|\underset{\sim}{\Sigma}^0_\alpha| = 2^{\aleph_0}$

for all α.

This implies the Thm:

$$|\mathcal{L}| = \left|\bigcup_{\alpha<\omega_1} \underset{\sim}{\Sigma}^0_\alpha\right| \leq \underset{[AC!]}{\aleph_1} \cdot 2^{\aleph_0} = 2^{\aleph_0}.$$

Induction proof.

$\underset{\sim}{\Sigma}^0_1$: every open set is of the form

$\bigcup_{i \in I} [s_i]$ where $\boxed{I \subseteq \mathbb{N}}$

and s_i is our enumeration of $\omega^{<\omega}$.

So $I \longmapsto \bigcup_{i\in I} [s_i]$ is a surjection from $\wp(\mathbb{N})$ onto $\underset{\sim}{\Sigma}^0_1$.

$\boxed{\text{So } |\underset{\sim}{\Sigma}^0_1| \leq 2^{\aleph_0}.}$

$\underset{\sim}{\Sigma} \longrightarrow \underset{\sim}{\Pi}$: Since $A \longmapsto \omega^\omega \setminus A$ is a bijection between $\underset{\sim}{\Sigma}^0_\alpha$ and $\underset{\sim}{\Pi}^0_\alpha$, we have

$$|\underset{\sim}{\Sigma}^0_\alpha| = |\underset{\sim}{\Pi}^0_\alpha|.$$

$\underset{\sim}{\Pi} \longrightarrow \underset{\sim}{\Sigma}$. Suppose for each $\alpha < \lambda$

$$|\underset{\sim}{\Pi}^0_\alpha| \leq 2^{\aleph_0}.$$

Using AC, pick surjections

$$S_\alpha : \omega^\omega \twoheadrightarrow \underset{\sim}{\Pi}^0_\alpha.$$

Define surjection S:

$$\boxed{\lambda^\omega \times \omega^\omega} \longrightarrow \underset{\sim}{\Sigma}^0_\lambda$$

$$((\alpha_i ; i \in \omega), x) \longmapsto \bigcup_{i \in \mathbb{N}} S_{\alpha_i}((x)_i)$$

This clearly is a surjection.

Since λ is countable, $|\lambda^\omega| = 2^{\aleph_0}$

So $|\lambda^\omega \times \omega^\omega| = 2^{\aleph_0}$.

q.e.d.

Notices of the American Mathematical Society 10 (1963), p. 593

63T-389. SOLOMON FEFERMAN, Stanford University, Stanford, California and AZRIEL LEVY, Hebrew University, Jerusalem, Israel. Independence results in set theory by Cohen's method, II.

Use the notation of the preceding abstract. Using a model of Cohen's type, one can prove that if ZF is consistent then it stays consistent after addition of the following axioms: (a) the set of real numbers is a denumerable union of denumerable sets, (b) every well-ordering of real numbers is finite or denumerable, (c) ω_1 is confinal with ω_0, (d) there is a Π^1_2 predicate $P(n,f)$ (where f,g denote variables ranging over n.t.f.'s) such that $(\forall n)(\exists f)P(n,f)$ but there is no sequence $\{\langle n,f_n\rangle \mid n < \omega\}$ of n.t.f.'s such that $(\forall n)P(n,f_n)$, (e) ω_1 is the ωth "constructible cardinal." (Received September 3, 1963.)

The Feferman-Lévy Model $\mathfrak{M}$

$\mathfrak{M} \models ZF +$ "$\mathbb{R}$ is a countable union of countable sets"

This implies:

The Axiom of Choice in the proof of $|\mathcal{B}| = 2^{\aleph_0}$ is not avoidable.

In FL model: $\mathbb{R} = \bigcup_{n \in \mathbb{N}} A_n$

So if $X \subseteq \mathbb{R}$: $X = \bigcup_{n \in \mathbb{N}} A_n \cap X$

$A_n \cap X \subseteq A_n$, so countable.

$\Longrightarrow$ $\mathfrak{M} \models$ every subset of $\mathbb{R}$ is a ctble union of ctble sets

$\Longrightarrow$ $\mathfrak{M} \models \mathcal{B} = \mathcal{P}(\mathbb{R})$.

The famous mistake of Henri Lebesgue

Measures are defined on σ-algebras $\mathfrak{A}$

In general $\mathfrak{A} \neq \mathcal{P}(\mathbb{R})$.

[Involves AC : Vitali set.]

Borel-σ-algebra ⟶ smallest σ-algebra containing the open sets.

Lebesgue : "All sets we care about are Borel."

He believed: $f : \mathbb{R} \longrightarrow \mathbb{R}$ is cts

$A \subseteq \mathbb{R}$ is Borel

$\overset{!!!}{\Longrightarrow}$ $f[A]$ is Borel

This is false !!

The mistake was spotted in 1917. Suslin proved that there are non-Borel sets which are cts images of Borel sets.

ANALYTIC SETS

The cts function here is even very simple. [projection]

Def Let $A \subseteq X^{n+1}$. Then we call

$$B = \{(x_1, \dots, x_n) ; \exists x \in X \; (x, x_1, \dots, x_n) \in A\}$$

the projection of A.

Write $pA := B$.

The map

$$\pi : (x, x_1, \dots, x_n) \longmapsto (x_1, \dots, x_n)$$

is clearly continuous, so

$$pA = \pi[A].$$

Thus projections are special cases of cts images.

Def. Let Γ be a pointclass. We define $\exists^{\omega^\omega}\Gamma$ a pointclass by

$$\exists^{\omega^\omega}\Gamma(X) := \{pA ; A \in \Gamma(\omega^\omega \times X)\}$$

We say Γ is closed under projections if

$$\exists^{\omega^\omega}\Gamma \subseteq \Gamma.$$

Suslin's Theorem now says: the pointclass BOREL is not closed under projections.

PROJECTIVE HIERARCHY

$$\boldsymbol{\Pi}^1_0(\omega^\omega) := \boldsymbol{\Pi}^0_1(\omega^\omega)$$

[If $X \neq \omega^\omega$, you might want to reconsider — $\boldsymbol{\Pi}^0_2$. ES #2.]

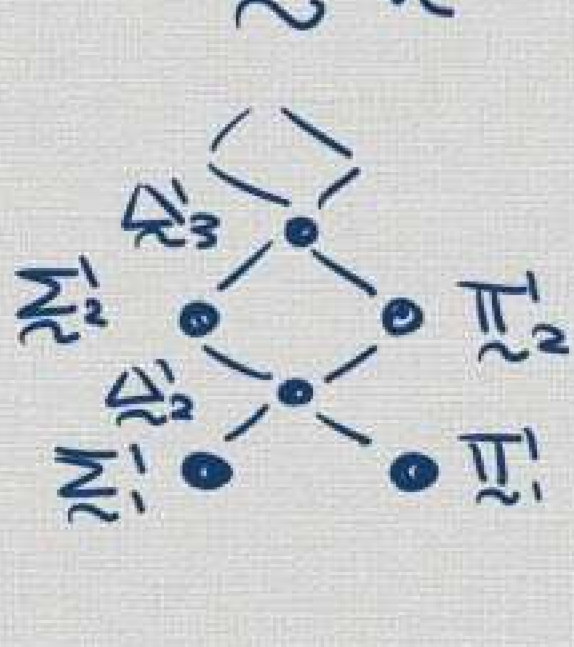
$$\boldsymbol{\Sigma}^1_{n+1} := \exists^{\omega^\omega}\boldsymbol{\Pi}^1_n$$

$$\boldsymbol{\Pi}^1_{n+1} := \breve{\boldsymbol{\Sigma}}^1_{n+1}.$$

$\boldsymbol{\Sigma}^1_1$ is called the ANALYTIC SETS.

$\boldsymbol{\Pi}^1_1$ is called the CO-ANALYTIC SETS.

$$\boldsymbol{\Delta}^1_n(X) := \boldsymbol{\Sigma}^1_n(X) \cap \boldsymbol{\Pi}^1_n(X).$$



Lebesgue believed that these are all contained in Le. HE WAS WRONG!

<u>Theorem</u> The projective hierarchy does not collapse.

<u>Pf.</u> Using the technique of universal sets. We know that $\underset{\sim}{\Pi}^1_0 = \underset{\sim}{\Pi}^0_1$ has a universal set. We know that if $\underset{\sim}{\Sigma}^1_n$ has a universal set, then $\underset{\sim}{\Pi}^1_n$ has a universal set.

So, left to show:
If V is universal for $\underset{\sim}{\Pi}^1_n$, find U universal for $\underset{\sim}{\Sigma}^1_{n+1}$.

Let $V \subseteq \underline{\omega^\omega} \times \underline{\omega^\omega} \times (\omega^\omega)^k$ be universal for $\underset{\sim}{\Pi}^1_n$.

$$U := \{(u, \vec{x}) \in \omega^\omega \times (\omega^\omega)^k \; ; \; \exists v \in \omega^\omega \; (u, v, \vec{x}) \in V\}$$

Clear that $U \in \underset{\sim}{\Sigma}^1_{n+1}(\omega^\omega \times (\omega^\omega)^k)$.

and the same x that is the code for the set A [in V] is the code for pA in U.

q.e.d.

Proposition Every Borel set is $\utilde{\Sigma}^1_1$.

[Example Sheet #2.]

Corollary Suslin' Thm:

there is a $\utilde{\Sigma}^1_1$ set that is not Borel.

Q. Does $\boxed{\mathrm{Det}(\utilde{\Sigma}^1_1)}$ hold?

$\utilde{\Sigma}^1_2$

$\utilde{\Sigma}^1_3$

$\vdots$

These are not ZFC theorems but closely connected to:

1. Large Cardinal Axioms
2. Definability of wellorders of ω^ω.