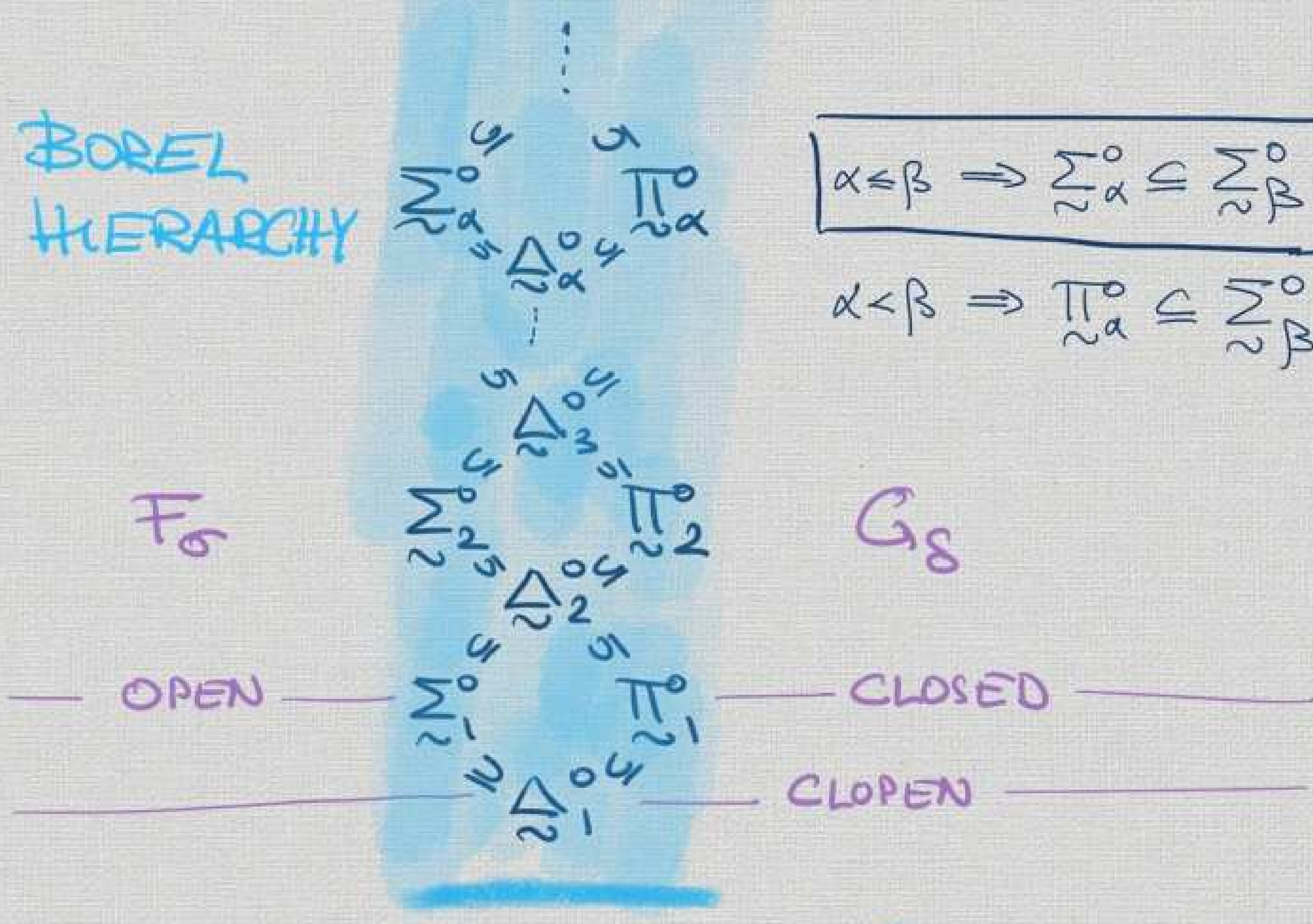
Infinite Games

Lent 2021
8 February 2021

Lecture VIII



$$\boxed{\alpha \le \beta \Rightarrow \mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Sigma}^0_\beta}$$

$$\alpha < \beta \Rightarrow \mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Sigma}^0_\beta$$

<u>Def.</u> A top. space is called a <u>G_δ space</u> if $\mathbf{\Pi}^0_1 \subseteq \mathbf{\Pi}^0_2$

"every closed set is a G_δ space".

<u>Note</u> Every metric space is a G_δ space.

For Cantor & Baire space, we proved that there is a ctble topology base of clopen sets and this implies being G_δ.

<u>Prop.</u> If X has a ctble clopen topology base, then X is G_δ.

<u>Proof.</u> Let $F \subseteq X$ be closed.

Let $G := X \setminus F$. [G is open]

For every $x \in G$, find B_x in the topology base s.t. $x \in B_x \subseteq G$.

Since B_x is clopen, $X \setminus B_x$ is also open.

$\{B_x ; x \in G\}$ is countable,

so let's write $\{B_n ; n \in \mathbb{N}\}$

Then $$F = \bigcap_{n \in \mathbb{N}} X \setminus B_n \in \underset{\sim}{\Pi}^0_2 .$$

[Since ctbl $\Rightarrow$ wellorderability, no choice needed.].

q.e.d.

By the Axiom of Replacement, the Borel Hierarchy terminates at some ordinal α [i.e., $\underset{\sim}{\Sigma}^0_\alpha = \underset{\sim}{\Pi}^0_\alpha$].

Q. What can we say about α?

Observation 1 If X is discrete, then every subset of X is clopen, so

$$\underset{\sim}{\Delta}^0_1 = \underset{\sim}{\Sigma}^0_1 = \underset{\sim}{\Pi}^0_1 .$$

Observation 2 If singletons are closed and X is countable, then

$$\underset{\sim}{\Delta}^0_2 = \underset{\sim}{\Sigma}^0_2 = \underset{\sim}{\Pi}^0_2 .$$

[If $A = \{x ; x \in A\}$ is ctble, then $A = \bigcup_{x \in A} \{x\}$.]

Better upper bound than by cardinality of X :

Proposition (ZFC).

For arbitrary X, $\underset{\sim}{\Delta}^0_{\aleph_1} = \underset{\sim}{\Sigma}^0_{\aleph_1} = \underset{\sim}{\Pi}^0_{\aleph_1}$.

Proof. It's enough to show

$$\underset{\sim}{\Sigma}^0_{\aleph_1} = \bigcup_{\alpha<\aleph_1} \Pi^0_\alpha .$$

Clearly: "$\supseteq$".

"$\subseteq$": If $A \in \underset{\sim}{\Sigma}^0_{\aleph_1}$, there are A_n s.t. $A = \bigcup_{n\in\mathbb{N}} A_n$ and $\alpha_n < \aleph_1$ s.t. $A_n \in \underset{\sim}{\Pi}^0_{\alpha_n}$

Since $\aleph_1$ is a regular cardinal, every ctble subset $A \subseteq \aleph_1$ is bounded, i.e., there is $\beta < \aleph_1$ s.t. $A \subseteq \beta$.

$$\{\alpha_n ; n\in\mathbb{N}\} \subseteq \aleph_1$$

Find bound $\underbrace{\beta}_{<\aleph_1}$: $\{\alpha_n ; n\in\mathbb{N}\} \subseteq \beta$

Then $\forall n$ $A_n \in \bigcup_{\alpha<\beta} \Pi^0_\alpha \Rightarrow A \in \underbrace{\underset{\sim}{\Sigma}^0_{\beta+1}}_{\subseteq \underset{\sim}{\Pi}^0_{\beta+2}}$

$\Longrightarrow$ CLAIM. q.e.d.

Height of the Borel hierarchy (in ZFC):

$$1 \leq \beta \leq \aleph_1.$$

<u>Theorem</u> (ZFC). If X is Cantor space, Baire space or $\mathbb{R}$, then the height of the Borel hierarchy is $\aleph_1$.

[In general, if X is uncountable Polish space.]

<u>This means</u>: if $\alpha < \aleph_1$, then $\underset{\sim}{\Sigma}^0_\alpha \neq \underset{\sim}{\Pi}^0_\alpha$.

The proof of Thm uses the <u>method of universal sets</u>.

<u>Def.</u> A <u>pointclass</u> is an operation that assign to each top. space X a set of subsets of X.

<u>Examples</u>: "open" / $\underset{\sim}{\Sigma}^0_1$

"closed" / $\underset{\sim}{\Pi}^0_1$

or $\underset{\sim}{\Sigma}^0_\alpha$ / $\underset{\sim}{\Pi}^0_\alpha$ / $\underset{\sim}{\Delta}^0_\alpha$

If Γ is a pointclass, we define

$$\check{\Gamma} \quad \text{by}$$

$$\check{\Gamma}(X) := \{ X \setminus A \,;\, A \in \Gamma(X) \}$$

called the dual pointclass of Γ

[Pronounced "Gamma dual".]

and

$$\Delta_\Gamma$$

by $\Delta_\Gamma(X) := \Gamma(X) \cap \check{\Gamma}(X)$

called the ambiguous pointclass of Γ.

Ex. $\underset{\sim}{\Pi}^0_\alpha$ is $\check{\underset{\sim}{\Sigma}}{}^0_\alpha$

$\underset{\sim}{\Delta}^0_\alpha$ is $\Delta_{\underset{\sim}{\Sigma}^0_\alpha}$ (and $\Delta_{\underset{\sim}{\Pi}^0_\alpha}$).

CLOSURE PROPERTIES OF POINTCLASSES:

For correctly chosen α and X.

	YES	NO
• closed under finite unions	$\underset{\sim}{\Sigma}^0_\alpha, \underset{\sim}{\Pi}^0_\alpha, \underset{\sim}{\Delta}^0_\alpha$	
• closed under finite intersections	$\underset{\sim}{\Sigma}^0_\alpha, \underset{\sim}{\Pi}^0_\alpha, \underset{\sim}{\Delta}^0_\alpha$	
• closed under ctble unions	$\underset{\sim}{\Sigma}^0_\alpha$	$\underset{\sim}{\Pi}^0_\alpha, \underset{\sim}{\Delta}^0_\alpha$
• closed under ctble intersections	$\underset{\sim}{\Pi}^0_\alpha$	$\underset{\sim}{\Sigma}^0_\alpha, \underset{\sim}{\Delta}^0_\alpha$
• closed under complements	$\underset{\sim}{\Delta}^0_\alpha$	$\underset{\sim}{\Sigma}^0_\alpha, \underset{\sim}{\Pi}^0_\alpha$

- Γ is <u>closed under cts preimages</u> if whenever $f : X \longrightarrow Y$ is cts and $A \in \Gamma(Y)$, then

BOLDFACE

$$f^{-1}[A] \in \Gamma(X).$$

- Γ is <u>closed under cts images</u> if whenever $f : X \longrightarrow Y$ is cts and $A \in \Gamma(X)$, then

$$f[A] \in \Gamma(Y).$$

A simple inductive argument shows that our Borel classes $\underset{\sim}{\Sigma}^0_\alpha, \underset{\sim}{\Pi}^0_\alpha, \underset{\sim}{\Delta}^0_\alpha$ are all <u>boldface</u>.

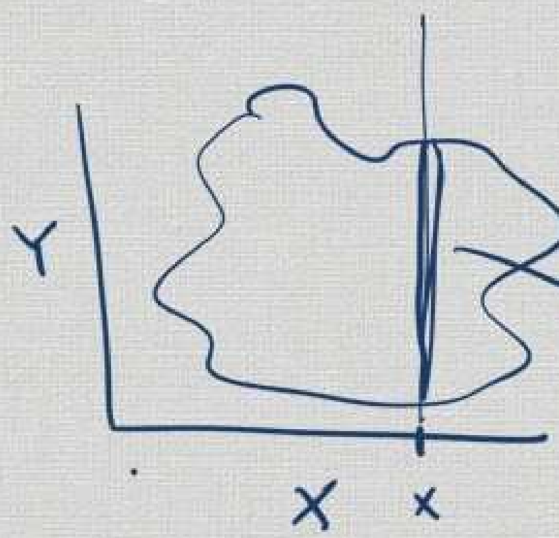
We're going to talk more about closure under cts images in Lectures $\underline{\text{IX}}$ & $\underline{\text{X}}$.

Definition Let X, Y be top. spaces and Γ a pointclass. A set $U \subseteq X \times Y$ is called **X-universal for $\Gamma(Y)$** if

① $U \in \Gamma(X \times Y)$ — product topology

② For every $A \in \Gamma(Y)$ there is some $x \in X$ s.t.

$$U_x = A.$$



$U_x := \{y \in Y \,;\, (x,y) \in U\}$ section of U at x.

Lemma Suppose U is X-universal for $\Gamma(X)$ and Γ is boldface. Then $\Gamma(X) \neq \check{\Gamma}(X)$. [non-selfdual]

Proof. Consider $X \times X \setminus U \in \check{\Gamma}(X \times X)$. and $x \longmapsto (x,x)$, $X \longrightarrow X \times X$ cts.

$$D := \{x \,;\, (x,x) \notin U\} \in \check{\Gamma}(X)$$

Assume $\Gamma(X) = \check{\Gamma}(X)$. Then find $d \in X$ s.t. $D = U_d$.

Then

$$d \in D \iff (d,d) \in U$$
$$\iff d \in U_d$$
$$\iff d \notin D. \quad \text{Contradiction.}$$

q.e.d.

Lecture IX.

We shall prove that for $\alpha < \aleph_1$, Σ^0_α has an ω^ω-universal set.

By Lemma, this implies our theorem.