INFINITE GAMES

Lecture VII
5 February 2021

Baire space ω^ω
Cantor space 2^ω

METRIC

$$d(x,y) := \begin{cases} 0 & \text{if } x=y \\ 2^{-m} & \text{if } x\restriction m = y\restriction m \ \& \ x(m) \neq y(m) \end{cases}$$

Open balls / basic open sets

$$[s] := \{x \in \omega^\omega \,;\, s \subseteq x\}$$

Countably many basic open sets.

Convergence

$$x_n \longrightarrow x$$

$$\iff \forall \varepsilon \exists N \forall n > N \ d(x_n, x) < \varepsilon$$

$$\iff \forall m \exists N \forall n > N \ d(x_n, x) < 2^{-m}$$

$$\iff \forall m \exists N \forall n > N \ x_n \restriction m = x \restriction m$$

If $A \subseteq \omega^\omega$, we can define

$$T_A := \{x\restriction n \,;\, x \in A, n \in \mathbb{N}\}$$
tree

Observe $A \subseteq \boxed{[T_A]}$:

$$\left[x \in A \longrightarrow \underset{\forall n \in \mathbb{N}}{x\restriction n \in T_A} \longrightarrow x \in [T_A] \right]$$

Proposition $[T_A]$ is the closure of A,

i.e., $\{x ; \exists (x_n)$ with $x_n \in A$ and $x_n \longrightarrow x\}$

Proof. Suppose $x_n \in A$ and $x_n \longrightarrow x$.

By our characterisation of convergence, this means that $x \upharpoonright k = x_n \upharpoonright k$ for some n,

so $x \upharpoonright k \in T_A$. Since k was arbitrary $x \in [T_A]$.

Suppose that $x \in [T_A]$. For every $k \in \mathbb{N}$, $x \upharpoonright k \in T_A$, so there is some $x_k \in A$ s.t. $x \upharpoonright k = x_k \upharpoonright k$.

Then by the characterisation of convergence: $x_k \longrightarrow x$.

So x is in the closure of A.

q.e.d.

Corollary $A \subseteq \omega^\omega$ is closed $\Longleftrightarrow$ $\exists T \; A = [T]$. (we know that $T := T_A$ does it).

TREE REPRESENTATION THM FOR CLOSED SETS

Some more topological properties:

Basic open sets $[s] = [T_s]$

where $T_s := \{t;\ s \subseteq t \text{ or } t \subseteq s\}$

So: basic open sets are closed

$\longrightarrow$ clopen

Spaces like this are called zero-dimensional.

If $x \in \omega^\omega$, then $\{x\} = [T_x]$ with

$T_x := \{x \upharpoonright n;\ n \in \mathbb{N}\}$

So singletons are closed and not open.

Hausdorff Separation (T_2):

If $x \neq y$ find n s.t. $x\upharpoonright n \neq y \upharpoonright n$.

Then $[x\upharpoonright n] \ni x$

$[y \upharpoonright n] \ni y$; $[x\upharpoonright n] \cap [y\upharpoonright n] = \emptyset$

CONTINUOUS FUNCTIONS [Proof on ES#2]

If $g : \omega^{<\omega} \longrightarrow \omega^{<\omega}$ s.t.

① g is order-preserving:

$s \subseteq t \longrightarrow g(s) \subseteq g(t)$

② g is "unbounded":

if $x \in \omega^\omega$ $\quad \mathrm{lh}(g(x\upharpoonright n)) \longrightarrow \infty$

$\hat{g}(x) := \bigcup_{n \in \mathbb{N}} g(x\upharpoonright n)$

if $g : \omega^{<\omega} \longrightarrow \omega^{<\omega}$ sat. (1) & (2)

$\hat{g} : \omega^{\omega} \longrightarrow \omega^{\omega}$

<u>Proposition</u> (w/o proof; ES#2)

$f : \omega^{\omega} \longrightarrow \omega^{\omega}$ is continuous iff there is $g : \omega^{<\omega} \longrightarrow \omega^{<\omega}$ with (1) & (2) s.t. $f = \hat{g}$.

<u>RULE OF THUMB</u>

A function f is continuous if and only if in order to determine $f(x)(k)$, I only need $x \upharpoonright n$ for some finite n.

Now consider functions from $(\omega^{\omega})^2$ to ω^{ω} and vice versa and ω^{ω} to ω^{ω}:

$$\left.\begin{array}{l} x \longmapsto x_{\mathrm{I}} \\ x \longmapsto x_{\mathrm{II}} \end{array}\right\} \text{ are all continuous}$$

$$(x, y) \longmapsto x * y$$

$$\begin{array}{l} x \longmapsto (x_{\mathrm{I}}, x_{\mathrm{II}}) \\ \omega^{\omega} \longrightarrow (\omega^{\omega})^2 \end{array}$$

Thus $(\omega^\omega)^2$ and ω^ω are homeomorphic by $(x,y) \longmapsto x*y$ with inverse $x \longmapsto (x_I, x_{II})$.

This phenomenon may explain the name zero-dimensional.

[Similarly: $(\omega^\omega)^k \cong (\omega^\omega)^\ell$ for any $k,\ell > 0$.

Theorem Baire space is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$.

[Proof uses continued fractions: If $x \in \mathbb{R}\setminus\mathbb{Q}$, then there is a sequence $a_i \in \mathbb{Z}^\omega$ s.t.

$$x = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}}$$]

So while topologically quite different from the reals $\mathbb{R}$, Baire space is set-theoretically very closed to the reals: many set-theoretic properties/proofs only depend on cardinality.

Example

ES#1 (4) has choice principles

$$AC_X(Y)$$

These are invariant under replacing X or Y with X', Y' s.t. X is in bij. with X' and Y is in bij. with Y'.

In particular,

$$AC_\omega(\mathbb{R}) \iff AC_\omega(\omega^\omega)$$
$$\iff AC_\omega(2^\omega)$$
$$\iff AC_\omega(X)$$

where X is any set in bij. w. $\mathbb{R}$.

IN SET THEORY, we often refer to elements of 2^ω or ω^ω as "reals" and abuse the notation by sometimes writing $\mathbb{R} := \omega^\omega$.

$\boxed{\text{MOTTO / HOPE}}$ If A is "simple", then A is determined.

Need a complexity hierarchy on Baire space:

Borel hierarchy

Let X be any topological space:

$$\underset{\sim}{\Sigma}^0_1 := \{A \subseteq X \,;\, A \text{ is open in } X\}$$

[BOLDFACE] SIGMA ZERO ONE

if $\underset{\sim}{\Sigma}^0_\alpha$ is defined

COMPLEMENTS

$$\underset{\sim}{\Pi}^0_\alpha := \{X \setminus A \,;\, A \in \underset{\sim}{\Sigma}^0_\alpha\}$$

if α is an ordinal & for all $\gamma < \alpha$, $\underset{\sim}{\Pi}^0_\gamma$ is defined

$$\underset{\sim}{\Sigma}^0_\alpha := \{A \,;\, \exists (A_n) \text{ s.t. for all } n\ A_n \in \bigcup_{\gamma<\alpha} \underset{\sim}{\Pi}^0_\gamma \text{ and } A = \bigcup_{n\in\mathbb{N}} A_n\}$$

CTBLE UNIONS

$$\underset{\sim}{\Delta}^0_\alpha := \underset{\sim}{\Sigma}^0_\alpha \cap \underset{\sim}{\Pi}^0_\alpha$$

Borel hierarchy

$$\utilde{\Sigma}^0_\alpha \supseteq \utilde{\Delta}^0_\alpha \subseteq \utilde{\Pi}^0_\alpha$$

$$\vdots$$

$$\begin{tikzcd}
 & & \utilde{\Delta}^0_3 \arrow[dr, phantom, sloped, "\supseteq"] & & \\
 & \utilde{\Sigma}^0_2 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\supseteq"] & & \utilde{\Pi}^0_2 & \\
 & & \utilde{\Delta}^0_2 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\supseteq"] & & \\
\text{OPEN} \arrow[r] & \utilde{\Sigma}^0_1 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[uu, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\supseteq"] & & \utilde{\Pi}^0_1 \arrow[uu, phantom, sloped, "\subseteq"] & \text{CLOSED} \arrow[l] \\
 & & \utilde{\Delta}^0_1 \arrow[ur, phantom, sloped, "\subseteq"] & & \text{CLOPEN} \arrow[ll]
\end{tikzcd}$$

<u>Properties</u> By definition $\boxed{\utilde{\Delta}^0_\alpha \subseteq \utilde{\Sigma}^0_\alpha, \utilde{\Pi}^0_\alpha}$

By definition $\alpha \leq \beta$, $\utilde{\Sigma}^0_\alpha \subseteq \utilde{\Sigma}^0_\beta$.

[So by complementation:
$\alpha \leq \beta$, $\utilde{\Pi}^0_\alpha \subseteq \utilde{\Pi}^0_\beta$.]

<u>Remains</u> $\alpha < \beta$, then $\boxed{\utilde{\Pi}^0_\alpha \subseteq \utilde{\Sigma}^0_\beta}$

$[\Leftrightarrow \utilde{\Sigma}^0_\alpha \subseteq \utilde{\Pi}^0_\beta]$

This follows from def. of $\utilde{\Sigma}^0_\beta$ by letting $A_n := A$, so $\bigcup_{n \in \mathbb{N}} A_n = A$.

<u>Q</u>: When does the Borel hierarchy terminate?

AFTER CLASS Question about REGULAR CARDINALS

Let λ be a limit ordinal. Then:

$$\mathrm{cf}(\lambda) := \min\{\alpha;\ \text{there is a cofinal fn } f:\alpha \longrightarrow \lambda\}$$

COFINALITY OF λ

unbounded

λ is **REGULAR** if $\lambda = \mathrm{cf}(\lambda)$

λ is **SINGULAR** if $\lambda > \mathrm{cf}(\lambda)$

These notions show up in L&ST, but not necessarily the words "regular" and "singular"

Theorem (Example Sheet in L&ST)

If $\alpha < \omega_1$, then there is no unbounded function $f: \alpha \longrightarrow \omega_1$.

[$\Longleftrightarrow$ ω_1 is regular]

NOTE THAT THIS THEOREM USES THE AXIOM OF CHOICE.

If you want to reformulate Example (4) without the word "regular", then (4) asks you to prove:

$$AC_\omega(\mathbb{R}) \Longrightarrow \text{every } f: \mathbb{N} \longrightarrow \aleph_1 \text{ is bounded}$$

$$\left[\text{i.e., } \forall f: \mathbb{N} \longrightarrow \aleph_1\ \exists \alpha < \aleph_1 \text{ s.t. } \mathrm{ran}(f) \subseteq \alpha\right]$$