INFINITE GAMES

Lecture VII
5 February 2021

Base space \( \omega \)
Countor space \( 2^\omega \)

Metric
\[
d(x,y) = \begin{cases} 
0 & \text{if } x = y \\
2^{-m} & \text{if } x(m) = y(m) \\
\infty & \text{if } x(m) \neq y(m)
\end{cases}
\]

Open balls / basic open sets
\[
[s] := \{ x \in \omega \mid s \subseteq x \}
\]

Countably many basic open sets.

Convergence
\[
x_n \rightarrow x
\]

\[\iff \forall N \exists n \forall x \in \omega \forall n > N \ d(x_n, x) < 2^{-m} \]

If \( A \subseteq \omega \), we can define
\[
T_A := \{ x \mid \forall n \in \mathbb{N} \ \exists x_n \in A, x_n \in \mathbb{N} \}
\]

Observe
\[
A \subseteq [T_A] :\]
\[
\left[ x \in A \rightarrow x \in \bigcap_{n \in \mathbb{N}} e[T_A] \rightarrow x \in [T_A] \right]
\]
Proposition \([T_A] \) is the closure of \(A\), i.e., \(\{x \mid \exists \{x_n\} \text{ with } x_n \in A \text{ and } x_n \to x\}\).

Proof. Suppose \(x_n \in A\) and \(x_n \to x\).

By our characterization of convergence, this means that \(x_n = x_n \in A\) for some \(n\), so \(x_n \in [T_A]\). Since \(n\) was arbitrary, \(x \in [T_A]\).

Suppose that \(x \in [T_A]\). For every \(k \in \mathbb{N}\), \(x_k \in [T_A]\), so there is some \(x_k \in A\) s.t. \(x_k = x_k \in A\).

Then by the characterization of convergence: \(x_k \to x\).
So \(x\) is in the closure of \(A\).

q.e.d.

Corollary. \(A \subseteq \omega^0\) is closed \(\iff \exists T \subseteq A = [T_A]\).

(We know that \(T := T_A\) does it.)

Tree representation

Thm for closed sets
Some more topological properties:

Basic open sets \([S] = [T_S]\)

where \(T_S := \{ t : j \text{ set or } t \in S \}\)

So: basic open sets are closed \(\implies\) clopen

Spaces like these are called zero-dimensional.

If \(x \in \omega^\omega\), then \(\{x\} = [T_x]\) with

\(T_x := \{ x \upharpoonright n : n \in \mathbb{N} \}\)

So singletons are closed and not open.

Turing Separation \((CT_2)\):

If \(x \neq y\) find \(n\) s.t. \(x \upharpoonright n \neq y \upharpoonright n\).

Then

\([x \upharpoonright n] \neq [y \upharpoonright n]\) \(\implies [x \upharpoonright n] \not\subseteq [y \upharpoonright n] \not\subseteq \emptyset\)

**Continuous Functions**

[Proof on ES#2]

If \(f : \omega^\omega \to \omega^\omega\) s.t.

1. \(f\) is order-preserving:
   \(x \leq y \implies f(x) \leq f(y)\)

2. \(f\) is "unbounded":
   \(\lim_{n \to \infty} f(x \upharpoonright n) = \infty\)
If \( g : \omega \to \omega \) is cat. \( \mathcal{Q} \mathcal{O} \mathcal{R} \),
\[ \exists \mathcal{C} : \omega \to \omega \]

**Proposition** (w/o proof; ES #2)

\( f : \omega \to \omega \) is continuous if there is \( g : \omega \to \omega \) with \( \mathcal{Q} \mathcal{O} \mathcal{R} \) s.t. \( f = g \).

**Rule of Thumb**

A function \( f \) is continuous if and only if in order to determine \( f(x)(k) \), I only need \( x \) for some finite \( x \).

Now consider functions from \( (\omega \times \omega)^2 \) to \( \omega \) and vice versa and \( \omega \) to \( \omega \):

\[ x \mapsto x_1 \mathcal{I} \quad \{ \text{are all continuous} \}
\]

\[ x \mapsto x_1 \mathcal{D} \]

\[ (x, y) \mapsto x \times y \]

\[ \omega \to \omega \]

\[ \omega \to (\omega \times \omega)^2 \]
Thus $(\omega^2)^2$ and $\omega^\omega$ are homeomorphic by $(x,y) \mapsto x \cdot y$ with inverse $x \mapsto (x_1, x_2)$. This phenomenon may explain the name zero-dimensional.

Similarly: $(\omega^k)^2 = (\omega^k)^2$ for every $k, k > 0$.

**Theorem**

Baire space is homeomorphic to $\mathbb{R}/\mathbb{Q}$.

[Proof uses continued fractions. If $x \in \mathbb{R}/\mathbb{Q}$, then there is a sequence $a_i \in \mathbb{Z}$ s.t.

$$x = \frac{1}{a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}}$$

So while topologically quite different from the reals $\mathbb{R}$, Baire space is set-theoretically very close to the reals: many set-theoretic properties/proofs only depend on cardinality.
Example

ES#1 (4) has chosen principles

$AC_\times(Y)$

These are invariant under replacing $X$ or $Y$ with $X', Y'$ s.t. $X$ is in bij. with $X'$ and $Y$ is in bij. with $Y'$.

In particular,

$AC_\omega(\mathbb{R}) \iff AC_\omega(\omega) \iff AC_\omega(2^\omega) \iff AC_\omega(X)$

where $X$ is any set in bij. $\omega$. $\mathbb{R}$

IN SET THEORY

we often refer to elements of $2^{\omega}$ as $\omega$ as "reals" and above the notation by sometimes writing $\bigtriangleup \mathbb{R} := \omega$. .
MOTTO/HOPE

If $A$ is "simple", then $A$ is determined.

Need a complexity hierarchy on base space:

**Borel hierarchy**

Let $X$ be any topological space:

$$
\{\Sigma_1^0 : = \{ A \subseteq X; A \text{ is open} \}
$$

**[Boldface] Sigma Zero One**

If $\Sigma_0^2$ is defined

$$
\Pi_1^0 := \{ X \setminus A; A \in \Sigma_0^2 \}
$$

If $\alpha$ is an ordinal and for all $\beta < \alpha$, $\Pi_0^\beta$ is defined

$$
\Sigma_0^\alpha := \{ A; \exists (A_n) \text{ s.t. for all } n \ A_n \in \Pi_0^{\alpha_n} \text{ and } A = \bigcup_{n \in \mathbb{N}} A_n \}
$$

**CTBLE Unions**

$$
\Delta_0^\alpha := \Sigma_0^\alpha \cap \Pi_0^\alpha
$$
Properties

By definition \( A_0 \prec \Sigma_0, \Pi_0 \)

By definition \( \alpha \sqsubseteq \beta, \; \Sigma_0^\alpha \prec \Sigma_0^\beta \).

[So by complementation:] \( \alpha \sqsubseteq \beta, \; \Pi_0^\alpha \prec \Pi_0^\beta \).

Remarks \( \alpha \sqsubset \beta \), then \( \Pi_0^\alpha \prec \Sigma_0^\beta \)

This follows from def. of \( \Sigma_0^\beta \) by letting \( A_n := A \), so \( \bigcup A_n = A \).

Q: When does the Borel hierarchy terminate?
Let $\lambda$ be a limit ordinal. Then:

$$\text{cf}(\lambda) := \min \{ \alpha ; \text{there is a cofinal function } f : \alpha \to \lambda \}$$

- $\alpha$ is **regular** if $\lambda = \text{cf}(\lambda)$
- $\alpha$ is **singular** if $\lambda > \text{cf}(\lambda)$

**Theorem** (Example Sheet in L&ST)

If $\alpha < \omega_1$, then there is no unbounded function $f : \alpha \to \omega_1$.

$[\iff \omega_1 \text{ is regular}]$

Note that this theorem uses the Axiom of Choice.

If you want to reformulate Example (4) without the word "regular", then (4) asks you to prove:

$$\text{AC}_{\omega}(\mathbb{R}) \rightarrow \text{ every } f : \mathbb{N} \to \omega_1 \text{ is bounded}$$

$[\left( \forall f : \mathbb{N} \to \omega_1 \right) \exists \alpha < \omega_1 \text{ s.t. } \text{ran}(f) \subseteq \alpha]$