INFINITE GAMES

Lecture VI

3 February 2021

FROM LECTURE V

Theorem (Gale–Stewart)
All closed sets $A \subseteq M^\omega$ are quasidetermined.

We are in the middle of the proof:

ℓ $\quad A = [T]$; $\ell(s) := \mathrm{II}$ if $s \notin T$

PARTIAL LABELLING $\ell : M^{<\omega} \longrightarrow \{\mathrm{I}, \mathrm{II}\}$

Given ℓ, $\ell^+ \supseteq \ell$ defined by

ℓ^+ if $\mathrm{lh}(s)$ is even and for all $m \in M$ $\ell(sm) = \mathrm{II}$, then
$\ell^+(s) := \mathrm{II}$

$\circledast$ if $\mathrm{lh}(s)$ is odd and for some $m \in M$ $\ell(sm) = \mathrm{II}$, then
$\ell^+(s) := \mathrm{II}$.

RECURSIVE DEFINITION $\quad \ell_0 := \ell$; $\ell_{\gamma+1} := \ell_\gamma^+$; $\ell_\lambda := \bigcup_{\gamma<\lambda} \ell_\gamma$

FIXED PT ℓ_α By Replacement, there is a fixed pt $\ell_{\alpha+1} = \ell_\alpha$.

$$\hat{\ell} : M^{<\omega} \longrightarrow \{\mathrm{I}, \mathrm{II}\} : \hat{\ell}(s) := \begin{cases} \mathrm{II} & \text{if } \ell_\alpha(s) = \mathrm{II} \\ \mathrm{I} & \text{o/w.} \end{cases}$$

FROM LECTURE V

Claim
If $\hat{\ell}(\emptyset) = \mathrm{I}$, then player I has a winning quasistrategy.
If $\hat{\ell}(\emptyset) = \mathrm{II}$, then player II has a w.q.s.

This claim proves the G-S Theorem.

<u>Subclaim 1</u> If $\hat{\ell}(\emptyset) = \underline{I}/\underline{II}$, define

$$Q_{\underline{I}} := \{ s \in \omega^{<\omega} ; \hat{\ell}(s) = \underline{I} \}$$

$$Q_{\underline{II}} := \{ s \in \omega^{<\omega} ; \hat{\ell}(s) = \underline{II} \}.$$

Then $Q_{\underline{I}} / Q_{\underline{II}}$ is a $\underline{I}/\underline{II}$-quasistrategic tree.

[Need to show:

① if $\hat{\ell}(s) = \underline{I}$ & $lh(s)$ is even then there is n s.t. $\hat{\ell}(sn) = \underline{I}$

② if $\hat{\ell}(s) = \underline{I}$ & $lh(s)$ is odd then for all n $\hat{\ell}(sn) = \underline{I}$

③ if $\hat{\ell}(s) = \underline{II}$ & $lh(s)$ is even then for all n $\hat{\ell}(sn) = \underline{II}$

④ if $\hat{\ell}(s) = \underline{II}$ & $lh(s)$ is odd then there is n s.t. $\hat{\ell}(sn) = \underline{II}$

We had $\ell_\alpha = \ell_{\alpha+1} = (\ell_\alpha)^+$, so ③ & ④ follow immediately from $\circledast$

Similarly, if

$lh(s)$ is even ① $\hat{\ell}(s) = \underline{I} \Longrightarrow s \notin \mathrm{dom}(\ell_\alpha)$

$\Longrightarrow$ there is n $sn \notin \mathrm{dom}(\ell_\alpha)$

$\Longrightarrow \hat{\ell}(sn) = \underline{I}$.

And ② is similar]

Subclaim 2 If $\hat{\ell}(\emptyset) = \underline{\mathrm{I}}$, then $Q_{\underline{\mathrm{I}}}$ is a w.q.s. for $\underline{\mathrm{I}}$.

[Need to show $[Q_{\underline{\mathrm{I}}}] \subseteq A$. So fix some $x \in [Q_{\underline{\mathrm{I}}}]$. So $\forall n$

$$x \restriction n \in Q_{\underline{\mathrm{I}}}$$
$$\Longrightarrow \hat{\ell}(x \restriction n) = \underline{\mathrm{I}}$$
$$\Longrightarrow \hat{\ell}(x \restriction n) \neq \underline{\mathrm{II}}$$
$$\Longrightarrow \ell(x \restriction n) \neq \underline{\mathrm{II}}$$
$$\Longrightarrow x \restriction n \in T$$

$\ell(s) = \underline{\mathrm{II}} \iff s \notin T$

So $x \in A$.]

Remark What we did so far, didn't really need trees / closed sets. Let

$$\ell : M^{<\omega} \longrightarrow \{\underline{\mathrm{II}}\} \text{ be any}$$

partial labelling. Do the transfinite recursion $\ell_{\alpha+1} := (\ell_\alpha)^+$, $\ell_0 := \ell$, ...

Find fixed pt ℓ_α; define $\hat{\ell}$; define $Q_{\underline{\mathrm{I}}}$. Then $Q_{\underline{\mathrm{I}}}$ is a q.s. that avoids all $s \in \mathrm{dom}(\ell)$.

For player II, it's not always the case that Q_{II} is winning

[cf. Example (12) on ES#1]

since you could stay on labels II without ever leaving the tree.

So, we need to work a bit harder.

Remember the age function:

$$\mathrm{age} : Q_{\mathrm{II}} \longrightarrow \alpha + 1$$

$$\mathrm{age}(s) := \text{the least } \beta \text{ s.t. } s \in \mathrm{dom}(\ell_\beta)$$

This means:

$$\mathrm{age}(s) = 0 \iff s \in \mathrm{dom}(\ell_0) = \mathrm{dom}(\ell)$$

$$\iff s \notin T.$$

Subclaim 3 If $\hat{\ell}(s) = \mathrm{II}$ and $lh(s)$ is even then $age(s) = 0$ or for all m $age(sm) < age(s)$.

If $\hat{\ell}(s) = \mathrm{II}$ and $lh(s)$ is odd then $age(s) = 0$ or there is m $age(sm) < age(s)$.

[Follows directly from $\circledast$.]

Define $\hat{Q}_{\mathrm{II}} \subseteq Q_{\mathrm{II}}$ by

$\emptyset \in \hat{Q}_{\mathrm{II}}$ and

$sm \in \hat{Q}_{\mathrm{II}} :\Longleftrightarrow age(sm) = 0$ or $sm \in Q_{\mathrm{II}}$ and $age(sm) < age(s)$

INFORMALLY Playing by $\hat{Q}_{\mathrm{II}}$ means: "play into positions labelled II reducing the age if you can".

If $\hat{\ell}(\emptyset) = \mathrm{II}$, then by Subclaim 3, Q_{II} is a q.s.

Subclaim 4 If $\hat{\ell}(\emptyset) = \mathrm{II}$, then $\hat{Q}_{\mathrm{II}}$ is winning for II:

$$[\hat{Q}_{\mathrm{II}}] \cap A = \emptyset.$$

[Suppose $x \in [\hat{Q}_{\mathrm{II}}]$. Consider

$$a_n := \mathrm{age}(x \upharpoonright n).$$

This is a strictly decreasing seq. of ordinals until it hits 0 by construction of $\hat{Q}_{\mathrm{II}}$. Since no infinite, strictly decr. seq. of ordinals exist, we find k s.t. $a_k = \mathrm{age}(x \upharpoonright k) = 0$.

$$\Longrightarrow x \upharpoonright k \notin T$$

$$\Longrightarrow x \notin A.\ ]$$

qed (Claim)

qed (GST)

Remark If M is wellorderable (e.g., $M = \mathbb{N}$), then GST says:

Every closed subset of M^ω is determined.

Also: Every complement of a closed set is determined.

[Follows directly from the proof.]

End of Lecture IV

MOTTO/HOPE: If a set is NICE or SIMPLE, then it is determined.

If NICE = CLOSED, then this is just GST.

Next goal: Define a topology on M^ω.

Let's focus on the case $M = \mathbb{N}$ or even $M = 2 = \{0,1\}$.

Definition $x, y \in \omega^\omega$

$$d(x,y) := \begin{cases} 0 & \text{if } x = y \\ 2^{-n} & \text{if } x \restriction n = y \restriction n \text{ and } x(n) \neq y(n) \end{cases}$$

This is a metric on ω^ω.

What are the open balls for this metric

$$B_\varepsilon(x) = \{ y \in \omega^\omega ; d(x,y) < \varepsilon \}$$

$\varepsilon = 2^{-n}$

$$B_{2^{-n}}(x) = \{ y \in \omega^\omega ; d(x,y) < 2^{-n} \}$$
$$= \{ y ; y \upharpoonright n+1 = \boxed{x \upharpoonright n+1} \}$$

OPEN BALLS ARE DETERMINED BY FINITE SEQ.

If $s \in \omega^{<\omega}$, write

$$[s] := \{ x \in \omega^\omega ; s \subseteq x \}$$

So $B_{2^{-n}}(x) = [x \upharpoonright n+1]$.

Thus the topology of the metric space is generated by the basic open sets

$$\{ [s]_{\omega^\omega} ; s \in \omega^{<\omega} \}.$$

This space is called BAIRE space; if we restrict to 2^ω, then it is called CANTOR space.

If you think of

ω^ω as $\prod_{i\in\omega} X_i$ with $X_i = \omega$

and 2^ω as $\prod_{i\in\omega} Y_i$ with $Y_i = 2$
$= \{0,1\}$

then Baire space is just the product topology on $\prod_{i\in\omega} X_i$ with the <u>discrete topology on ω</u>;

and Cantor space is the product topology on $\prod_{i\in\omega} Y_i$ with the <u>discrete topology on 2</u>.

Tychonoff implies that Cantor space is compact, but Baire space is not.

Easy to see the latter:

$$\omega^\omega = \bigcup_{n\in\omega} \underbrace{[\langle n\rangle]}_{=\{x;\, x(0)=n\}}$$

<u>Next time:</u>

Show that $A = [T] \subseteq \omega^\omega$

$\Longleftrightarrow$

A is closed in Baire space.