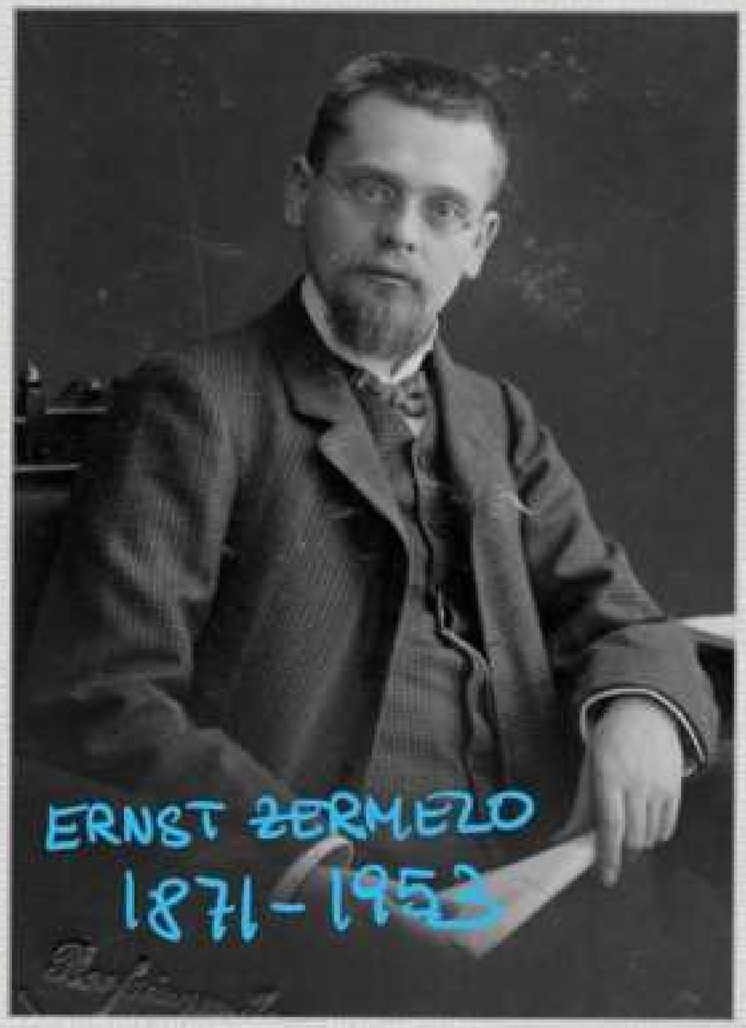
INFINITE GAMES

Lent 2021
1 FEBRUARY 2021

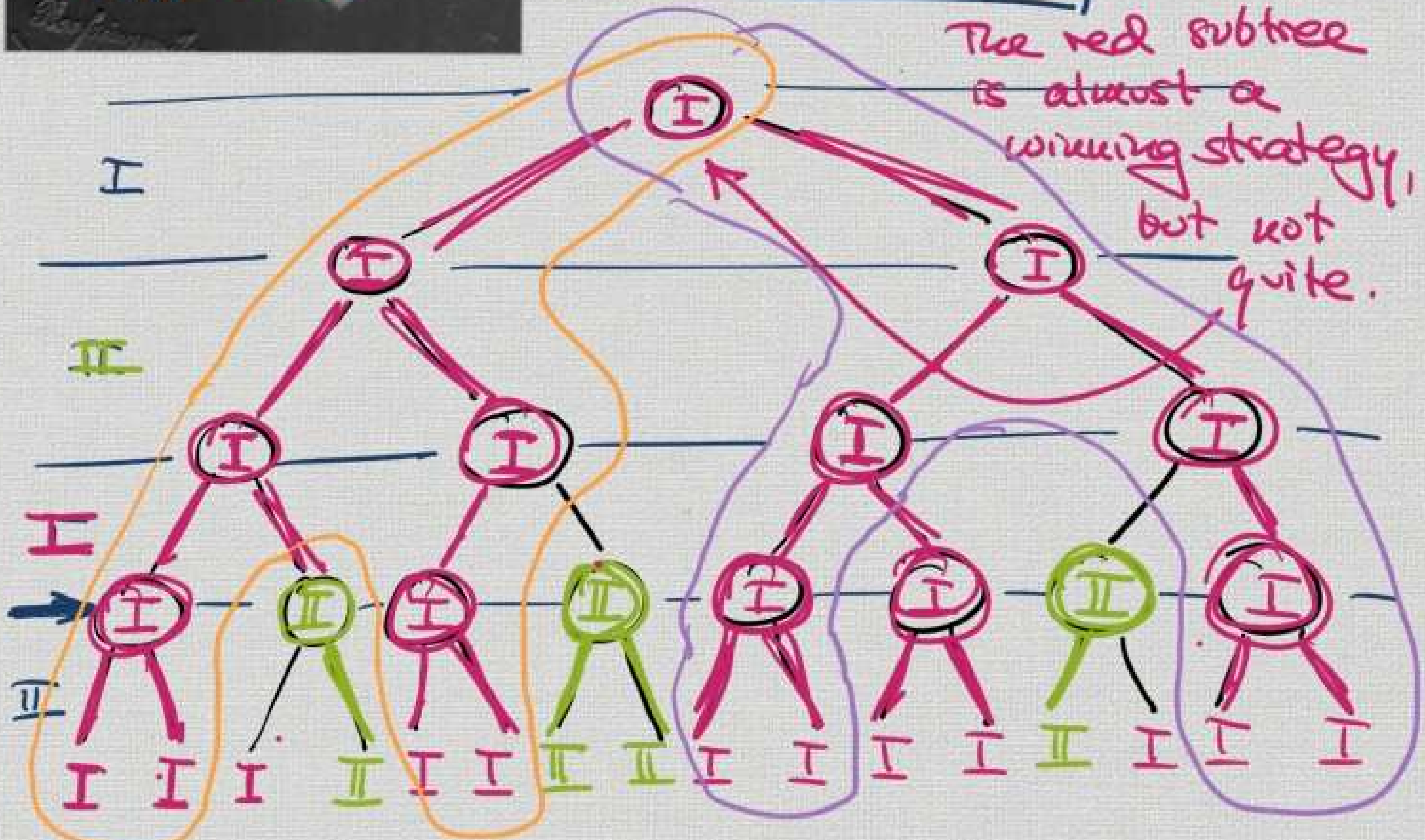
LECTURE V



ZERMELO'S THEOREM

Every finite game is determined.

BACKWARDS INDUCTION



Definition A function
$$\sigma : M^{<\omega} \longrightarrow \wp(M) \setminus \{\emptyset\}$$
is called a quasistrategy

Strategies can be identified as a special case of quasistrategies:
$$\forall s \quad |\sigma(s)| = 1.$$
Quasistrategic trees:
$$Q_\sigma^{I} := \{ s \in M^{<\omega} ;\ \forall n \quad s(2n) \in \sigma(s \upharpoonright 2n) \}.$$
$$Q_\sigma^{II} := \{ s \in M^{<\omega} ;\ \forall n \quad s(2n+1) \in \sigma(s \upharpoonright 2n+1) \}$$
A quasistrategy is winning for I in $G(A)$ if
$$[Q_\sigma^{I}] \subseteq A$$
and winning for II in $G(A)$ if
$$[Q_\sigma^{II}] \cap A = \emptyset.$$
As before, at most one of the two players can have a winning quasistrategy.

Definition A set $A \subseteq M^\omega$ is called **quasidetermined** if either I or II has a winning qs in $G(A)$.

Lemma If M is wellorderable, then every quasistrategic tree on M contains a strategic tree on M.

[**Conseq.** If M is wellorderable and $A \subseteq M^\omega$ is quasidetermined, then it is determined.]

Proof. Let σ be a quasistrategy

$$\sigma : M^{<\omega} \longrightarrow \wp(M) \setminus \{\emptyset\}$$

If M is wellorderable, there is a choice function

$$c : \wp(M) \setminus \{\emptyset\} \longrightarrow M$$

$$[\text{s.t. } c(A) \in A]$$

$$\sigma^* := c \circ \sigma : M^{<\omega} \longrightarrow M$$

By construction, $T^{\mathrm{I}}_{\sigma^*} \subseteq Q^{\mathrm{I}}_{\sigma}$ and $T^{\mathrm{II}}_{\sigma^*} \subseteq Q^{\mathrm{II}}_{\sigma}$. q.e.d.

Compare our proof of the existence of a non-determined set from L $\underline{IV}$

[we used a wellordering of M^ω to get $A \subseteq M^\omega$ non-determined]

to trees, where we use a wellordering of M.

<u>NOTICE</u>: This one uses <u>less choice</u> than the other one:

The most interesting example is $M = \mathbb{N}$ (obviously wellordered), but then $M^\omega = \omega^\omega$ is not obviously wellordered.

<u>Definition</u> A set $A \subseteq M^\omega$ is called <u>closed</u> if there is a tree T on M s.t.

$$A = [T].$$

<u>Remark 1</u> This is actually the notion of being closed in a topological space: cf. Lecture $\underline{VI}$.

Remark 2 Zermelo's finite games can be represented by closed payoff sets.

Finite game of length n:
Let's say $f : M^n \longrightarrow \{I, II\}$ is the function labelling the leaves according to which player wins.

$$A := \{ x \in M^\omega \, ; \, f(x \upharpoonright n) = I \}$$

Then the finite game is just $G(A)$.

$$T := \{ s \in M^{<\omega} \, ; \, \begin{array}{l} f(s \upharpoonright n) = I \\ \text{and } lh(s) \geq n \\ \text{or } lh(s) < n \text{ and there is} \\ \text{a } t \supseteq s \text{ s.t. } f(t) = I \end{array} \}$$

Clearly, $[T] = A$.

And thus all finite games are closed games.

Theorem (Gale-Stewart)

All closed sets $A \subseteq M^\omega$ are quasidetermined.

Proof. if $A = [T]$.

This means that if $x \notin A$, then

$$x \notin [T] = \{x \in M^\omega ; \forall n \ x \upharpoonright n \in T\}$$

$\Longrightarrow$ there is some n s.t.

$$x \upharpoonright n \notin T.$$

These are positions won for sure by player II.

PARTIAL LABELLINGS

Define a partial function

$$\ell : M^{<\omega} \longrightarrow \{\mathrm{I}, \mathrm{II}\}$$

by $\ell(s) = \mathrm{II}$ iff $s \notin T$.

Apply the following recursion rules to the partial labellings:

RECURSION RULES

$$\boxed{\ell \rightsquigarrow \ell^+} \supseteq \ell$$

extended partial labelling

if $\ell(s)$ is undefined

- and $\ell h(s)$ is even [I moves]
  and $\forall m \in M \;\; \ell(sm) = \mathrm{II}$,
  then $\ell^+(s) := \mathrm{II}$
- and $\ell h(s)$ is odd [II moves]
  and $\exists m \in M \;\; \ell(sm) = \mathrm{II}$,
  then $\ell^+(s) := \mathrm{II}$.

TRANSFINITE RECURSION:

$$\ell_0 := \ell$$

$$\ell_{\alpha+1} := (\ell_\alpha)^+$$

$$\ell_\lambda := \bigcup_{\alpha<\lambda} \ell_\alpha$$

Since $\ell^+ \supseteq \ell$, this is a partial function.

It's easy to construct examples of truly transfinite processes like this [needs $|M| \geqslant \aleph_0$].

<u>Claim</u> This process terminates at some ordinal α, i.e.,

$$\ell_\alpha = \ell_{\alpha+1} = (\ell_\alpha)^+ .$$

[<u>Proof.</u> For $s \in M^{<\omega}$, we can define an <u>age function</u>:

$$\mathrm{age}(s) := \begin{cases} \text{least } \beta \text{ s.t. } \ell_\beta(s) \text{ is defined} \\ 0 \quad \text{if } \ell_\beta(s) \text{ is never defined.} \end{cases}$$

So if the process never terminates, then age is a surjection from the set $M^{<\omega}$ onto the (proper) class Ord. This contradicts the Axiom of Replacement.]

Let α be this termination point

$$\hat{\ell}(s) := \begin{cases} \mathrm{I\!I} & \text{if } s \in \mathrm{dom}(\ell_\alpha) \\ \mathrm{I} & \text{o/w.} \end{cases}$$

$\hat{\ell} \geq \ell_\alpha$. $\hat{\ell}$ is a total function.

Claim

If $\hat{\ell}(\emptyset) = \mathrm{I}$, then player I has a winning quasistrategy.

If $\hat{\ell}(\emptyset) = \mathrm{II}$, then player II has a w.q.s.

[This proves the G-S theorem.]

In lecture VI, we'll prove the above Claim.