Infinite Games

Lecture IX

10 February 2021

Borel Hierarchy Theorem
On Baire space, the height of the Borel hierarchy is exactly $\aleph_1$.

- i.e., for $\alpha < \omega_1$

$$\underset{\sim}{\Sigma}^0_\alpha \neq \underset{\sim}{\Pi}^0_\alpha$$

Definition A set $U \subseteq X \times Y$ is called X-universal for $\Gamma(Y)$ if $U \in \Gamma(X \times Y)$ and for each $A \in \Gamma(Y)$ there is $x \in X$ s.t. $A = U_x$.

Lemma If there is an X-universal set for $\Gamma(X)$, then $\Gamma(X) \neq \check{\Gamma}(X)$. $\boxed{\text{PROVED IN LECTURE VIII}}$

Work in ZFC and check uses of AC in the proof later, after we're done.

Proof by induction:

L1 Induction base $\underset{\sim}{\Sigma}^0_1$

L2 Complementation step: $\underset{\sim}{\Sigma} \longrightarrow \underset{\sim}{\Pi}$

L3 Ctble union step: $\underset{\sim}{\Pi} \longrightarrow \underset{\sim}{\Sigma}$.

<u>Lemma 1</u> There is a ω^ω-universal set for $\underset{\sim}{\Sigma}^0_1(\omega^\omega)$.

<u>Proof.</u> An open set is an arb. union of basic open sets. [Since there are only ctbly many of those: ctble union.]

$$P = \bigcup_{i \in I} [s_i]$$ where I is a ctble index set.

Let $\{s_i ; i \in \mathbb{N}\}$ be your favourite enumeration of $\omega^{<\omega}$. Thus $\{[s_i] ; i \in \mathbb{N}\}$ is an enum. of the b.o.s.

Define $U := \{(x,y) ; \exists i \in \mathbb{N}$
$x(i) \neq 0 \wedge s_i \subseteq y\}$

Then U is universal for $\underset{\sim}{\Sigma}^0_1(\omega^\omega)$.

(1) U is open: if $(x,y) \in U$ then there is $i \in \mathbb{N}$ $x(i) \neq 0$ and $s_i \subseteq y$

Define $t := x \upharpoonright i+1$.

Then $[t, s_i] \subseteq U$.

$\{(v,w) ; t \subseteq v$ and $s_i \subseteq w\}$

(2) If P is open, then let

$$x(i) := \begin{cases} 1 & \text{if } [s_i] \subseteq P \\ 0 & \text{if } [s_i] \not\subseteq P \end{cases}$$

Then $$P = \bigcup_{x(i) \neq 0} [s_i]$$

Thus $P = U_x$. q.e.d.

LEMMA 2 If $U \subseteq X \times X$ is X-universal for $\Gamma(X)$, then $X \times X \setminus U$ is X-universal for $\check{\Gamma}(X)$.

Pf. Obvious.

LEMMA 3 Let $\lambda < \omega_1$. Suppose that for each $\alpha < \lambda$, there is an ω^ω-universal set U_α for $\underset{\sim}{\Pi}^0_\alpha(\omega^\omega)$. Then there is an ω^ω-universal set for $\underset{\sim}{\Sigma}^0_\lambda$.

Proof. If λ is a successor ordinal, i.e., $\lambda = \mu + 1$, then let $\alpha_n := \mu$ for all n.

If λ is limit, pick a seq. α_n s.t. $\bigcup \alpha_n = \lambda$.

Observe that if

$$A = \bigcup_{n\in\mathbb{N}} A_n \quad \text{where } A_n \in \bigcup_{\alpha<\lambda} \underset{\sim}{\Pi}^0_\alpha$$

then I find (by "postponing if necessary") a sequence $A_n' \in \underset{\sim}{\Pi}^0_{\alpha_n}$ s.t.

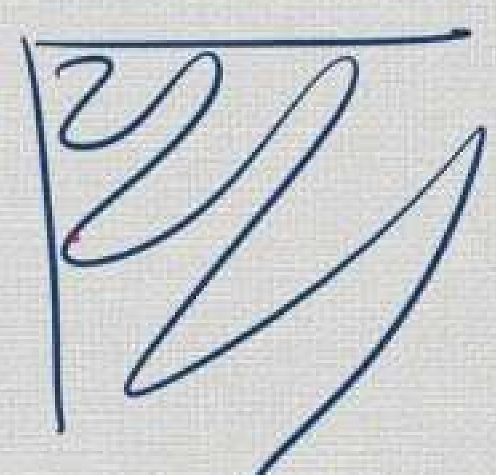
$$\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} A_n' .$$ (*)

(*) To simplify notation, write $U_n := U_{\alpha_n}$.

Encoding ctble seq. of ω^ω by an element of ω^ω:

Take favourite bijection

$$\ulcorner \cdot,\cdot \urcorner : \omega\times\omega \longrightarrow \omega$$



If $x\in\omega^\omega$ and $n\in\mathbb{N}$, define $(x)_n \in \omega^\omega$ by

$$(x)_n(m) := x(\ulcorner n,m \urcorner)$$

$$x \longmapsto ((x)_n ; n\in\mathbb{N})$$

is a bijection between ω^ω & $(\omega^\omega)^\omega$.

Note also that $x \longmapsto (x)_n$ is continuous.

$$U := \{(x,y)\,;\ \exists n\ \ (((x)_n, y) \in U_n\}$$

<u>Claim</u> U is ω^ω-universal for $\underset{\sim}{\Sigma}^0_\lambda$.

(1) $$\overline{U_n} := \{(x,y)\,;\ ((x)_n, y) \in U_n\}$$

is the preimage of U_n under the cts map $(x,y) \longmapsto ((x)_n, y)$, so it's in $\underset{\sim}{\Pi}^0_{\alpha_n}$ [closure of the Borel classes under cts preimages].

But $U = \bigcup_{n\in\mathbb{N}} \overline{U_n}$, so

U is $\underset{\sim}{\Sigma}^0_\lambda$.

(2) Let A be $\underset{\sim}{\Sigma}^0_\lambda$. By (*), we

(**) find $A_n \in \underset{\sim}{\Pi}^0_{\alpha_n}$ s.t. $A = \bigcup_{n\in\mathbb{N}} A_n$.

By universality of U_n, we find x_n s.t. $A_n = (U_n)_{x_n}$.

Now fold up the seq. $(x_n;\ n\in\omega)$ into a single elt * of ω^ω s.t. $(x)_n = x_n\ \forall n$.

$$\left[x(\ulcorner n,m \urcorner) := x_n(m) \right]$$

$$\left[\begin{array}{l} A_n = (U_n)_{x_n} \;\textcircled{1}, \quad (x)_n = x_n \;\textcircled{2}, \quad A = \bigcup_{n\in\mathbb{N}} A_n \;\textcircled{3}, \\ U = \{(x,y);\ \exists n\ ((x)_n, y) \in U_n\} \;\textcircled{4} \end{array} \right]$$

<u>Claim</u> $U_x = A$

$$\begin{aligned} y \in A &\overset{\textcircled{3}}{\Longleftrightarrow} \exists n \ \ y \in A_n \\ &\overset{\textcircled{1}}{\Longleftrightarrow} \exists n \ \ y \in (U_n)_{x_n} \\ &\Longleftrightarrow \exists n \ \ (x_n, y) \in U_n \\ &\overset{\textcircled{2}}{\Longleftrightarrow} \exists n \ \ ((x)_n, y) \in U_n \\ &\overset{\textcircled{4}}{\Longleftrightarrow} (x,y) \in U \\ &\Longleftrightarrow y \in U_x. \end{aligned}$$

q.e.d.

<u>COROLLARY</u> Borel Hierarchy Theorem

<u>Pf.</u> This is an induction proof using L1–L3.

(***) [<u>Recursive def.</u> of a ω^ω-universal set for $\underset{\sim}{\Sigma}^0_\lambda$ for each $\lambda < \omega_1$.]

REMARK

Let's check how much choice we used:

Lemma 1 is a ZF theorem.

Lemma 2 is a ZF theorem.

Lemma 3 $\lambda \longmapsto$ pick α_n s.t. $\bigcup \alpha_n = \lambda$

[If λ is fixed, no choice fn needed.]

(*) picks from the nonempty sets

$W_\alpha := \{U;\ U$ is universal for $\underset{\sim}{\Pi}^0_\alpha\}$

an element.

(**) If $A \in \underset{\sim}{\Sigma}^0_\lambda$, then

$S_A := \{(A_n)_n;\ A = \bigcup_{n\in\mathbb{N}} A_n\} \neq \emptyset$

for every such A, so we use a choice fn for this family.

Corollary uses choice to pick for each limit $\lambda < \omega_1$ a seq. α_n in order to apply L3.

BACK TO GAMES

If Γ is a pointclass, write

$\mathrm{Det}(\Gamma)$ for

"$\forall A \in \Gamma(\omega^\omega)$, A is determined".

G-S proved $\mathrm{Det}(\underset{\sim}{\Pi}^0_1)$

The proof implies $\mathrm{Det}(\underset{\sim}{\Sigma}^0_1)$.

ES #1 (11): In general, the class of determined sets is not closed under complementation.

ES #1 (10): If a pcl. Γ is closed under the operation

$$A \longmapsto \{nx ; x \in A\}$$

then $\mathrm{Det}(\Gamma) \Longrightarrow \mathrm{Det}(\check{\Gamma})$.

Borel pointclasses are closed under this operation.

WOLFE (1955) Det($\underset{\sim}{\Sigma}^0_2$)

Pacific Journal of Mathematics

PJM 5:5 (1955), 841-847

THE STRICT DETERMINATENESS OF CERTAIN INFINITE GAMES

PHILIP WOLFE

1. Introduction. Gale and Stewart [1] have discussed an infinite two-person game in extensive form which is the generalization of a game as defined by Kuhn [3] obtained by deleting the requirement of finiteness of the game tree and regarding as plays all unicursal paths of

DAVIS (1964) Det($\underset{\sim}{\Sigma}^0_3$)

ADVANCES
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INFINITE GAMES OF PERFECT INFORMATION

Morton Davis

§ 1. INTRODUCTION

It is well-known that finite, two-person, zero-sum games with perfect information are strictly determined [1]. There have been attempts to remove from this result each of these restrictions. It is the first of these with which we will be concerned, i.e., we consider infinite games.

In a paper by Gale and Stewart [2], zero-sum, two-person, infinite games with perfect information are defined. Familiarity with this paper will be assumed.

The notation of this paper will lean heavily on the above paper, but some additions and modifications will be made. In referring to the game Γ we will mean the $(X_0, X_I, X_{II}, X, f, S, S_I, S_{II})$ of Gale and Stewart, where each element is understood as given in the game. We will also write $\Gamma(S_I = A)$ to stand for the game $(X_0, X_I, X_{II}, X_e, f, S, A, A^c)$ where $S = A \cup A^c$. We use here and elsewhere in the paper the superscript c to denote complement. We assume $f^{-1}(x)$ is always a finite set.

In this paper we extend the results of Gale and Stewart [2] and Wolfe [3], answer Questions 1 and 2 of [2] (assuming the Continuum Hypothesis), and finally, characterize the winning sets of a game suggested to me by

PARIS (1972) Det($\underset{\sim}{\Sigma}^0_4$)

THE JOURNAL OF SYMBOLIC LOGIC
Volume 37, Number 4, Dec. 1972

ZF ⊢ Σ^0_4 DETERMINATENESS

J. B. PARIS

Introduction. In this paper we show that in Zermelo-Fraenkel set theory (ZF) Σ^0_4 sets of reals are determinate.

Before proceeding to the proof it will be helpful to consider some previous work in this area. The first major result was obtained by Gale and Stewart [3] who showed that in ZF open games are determinate. This was then successively improved by Wolfe [4] to Π^0_2 (and so of course Σ^0_2) and then by Morton Davis [1] to Π^0_3. The results of Morton Davis further showed that countable unions of sufficiently 'simple' determinate sets are also determinate. At this time, however, Π^0_3 sets did not appear sufficiently simple for this method to be applied in order to get Σ^0_4 determinacy.

ANNALS OF MATHEMATICAL LOGIC – Volume 2, No. 3 (1971) pp. 325–357

HIGHER SET THEORY AND MATHEMATICAL PRACTICE *

Harvey M.FRIEDMAN
Stanford University

Received 17 April 1970

Introduction

When we examine the classical set-theoretic foundations of mathematics, we see that the only sets that play a role are sets of restricted type; at the risk of understatement, only sets of rank $< \omega + \omega$. Further examination reveals four fundamental principles about sets used: the existence of an infinite set; the existence of the power set of any set; every property determines a subset of any set; and the axiom of choice.

Proof-theoretic paper:
You cannot prove Det($\underset{\sim}{\Sigma}^0_5$) without using iterations of the power set axiom.

Annals of Mathematics, **102** (1975), 363-371

Martin (1975): Det(Borel)

Borel determinacy

BY DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initial segment (including the empty one) of an element of Y belongs to Y and such that every element of Y is a proper initial segment of an element of Y. Let $\mathcal{F}(Y)$ be the collection of all infinite sequences $\langle y_0, y_1, \cdots \rangle$ all of whose finite initial segments belong to Y. For each $A \subseteq \mathcal{F}(Y)$ we define a two person game of perfect information $\mathcal{G}(A, Y)$. Two players, I and II, take turns moving: I picks y_0, with $\langle y_0 \rangle \in Y$, II picks y_1 with $\langle y_0, y_1 \rangle \in Y$, I picks y_2 with $\langle y_0, y_1, y_2 \rangle \in Y$, etc. I *wins* just in case $\langle y_i : i \in \omega \rangle \in A$. ($\omega$ = the set of all natural numbers.) A *strategy for* I is a function s with domain the set of all elements