INFINITE GAMES

Lecture IV
29 Jan 2021

$M = \mathbb{N}$

NECESSARY I wins $G(A) \longrightarrow |A| = 2^{\aleph_0}$
II wins $G(A) \Longrightarrow |\omega^\omega \setminus A| = 2^{\aleph_0}$

SUFFICIENT If A is countable, then player II wins.

WARM-UP

[we write "I/II wins" as shorthand for "I/II has a w.s.".]

Theorem ① If $A \subseteq \omega^\omega$ s.t. $|A| < 2^{\aleph_0}$, then player II has w.s. in $G(A)$.

② If $A \subseteq \omega^\omega$ s.t. $|\omega^\omega \setminus A| < 2^{\aleph_0}$, then player I has w.s. in $G(A)$.

Proof. Proofs of ① & ② are essentially just switching the roles of I, II. So, we're going to prove ①.

CAUTION This type of symmetry argument is a major source of error: our game are NOT symmetric, so one needs to be extra careful with statements like these.

Let $\boxed{|A| < 2^{\aleph_0}.}$

Define an eq. relation $\sim$ on ω^ω by

$$x \sim y :\iff x_{\mathrm{II}} = y_{\mathrm{II}}$$

So, equivalence classes look like this

$$C_z := \{ x \,;\, x_{\mathrm{II}} = z \}$$

so there is a bijection between the $\sim$-eq. classes and ω^ω. Thus, there are $2^{\aleph_0}$ many equivalence classes.

By PHP, we find z s.t.

$$C_z \cap A = \emptyset.$$

Define τ as

$$\boxed{\text{IGNORE EVERYTHING PLAYER I DOES AND JUST PLAY THE NEXT DIGIT OF } z}$$

$$\tau(s) := z(n) \quad \text{if } \mathrm{lh}(s) = 2n+1$$

If σ is any strategy, then $(\sigma * \tau)_{\mathrm{II}} = z$.

$\Longrightarrow \quad \sigma * \tau \in C_z \qquad [C_z \cap A = \emptyset]$

$\Longrightarrow \quad \sigma * \tau \notin A.$

So τ is a w.s. for $\underline{\mathrm{II}}$.

q.e.d.

<u>BLINDFOLDED STRATEGIES</u>

USUAL NOTATION

There is $z \in \omega^\omega$ s.t.

τ_z for player $\underline{\mathrm{II}}$ $\tau(s) := z(n)$ if $lh(s) = 2n+1$

σ_z for player $\underline{\mathrm{I}}$ $\sigma(s) := z(n)$ if $lh(s) = 2n$

<u>Consequence</u> If A is not determined, then $|A| = |\omega^\omega \setminus A| = 2^{\aleph_0}$.

<u>Next goal</u>: Find such a non-determined set.

<u>Theorem</u> (AC). There is a non-determined subset $A \subseteq \omega^\omega$.

<u>Proof.</u> We did prove in L $\underline{\mathrm{III}}$ that if T is a strategic tree, then $|[T]| = 2^{\aleph_0}$.

<u>Q</u> How many strategic trees are there?

$\mathrm{Trees} := \{T \,;\, T \text{ is a tree on } \omega\}$

$\mathrm{STrees} := \{T \,;\, T \text{ is a strategic tree on } \omega\}$.

$T \subseteq \underbrace{\omega^{<\omega}}_{\text{COUNTABLE}}$ so, this gives an upper bound on the size of Trees

$$\boxed{2^{\aleph_0} \leq |\mathrm{STrees}| \leq |\mathrm{Trees}| \leq 2^{\aleph_0}}$$

if $z, z' \in \omega^\omega$, $z \neq z'$, then $[T^{I}_{\sigma_z}] \cap [T^{I}_{\sigma_{z'}}] = \emptyset$

$$\Longrightarrow T^{I}_{\sigma_z} \neq T^{I}_{\sigma_{z'}}$$

Together (with Cantor-Schröder-Bernstein), we get a bijection between $2^{\aleph_0}$ and STrees.

REMARK So far we didn't use any choice.

We are not going to use full AC, but only that the set ω^ω is wellorderable. This implies:

- $2^{\aleph_0}$ is an ordinal [so we can do transfinite recursion on it]
- there is a choice fn $c : \wp(\omega^\omega) \setminus \{\emptyset\} \longrightarrow \omega^\omega$ [i.e., $c(A) \in A$]

We had that $|\mathrm{STrees}| = 2^{\aleph_0}$, so

write $\mathrm{STrees} = \{T_\alpha ; \alpha < 2^{\aleph_0}\}$

and do the following transfinite recursion:

We're going to define A_α, B_α for $\alpha < 2^{\aleph_0}$.

s.t. $|A_\alpha| = |B_\alpha| = |\alpha|$. (*)

we will see $A_\alpha \cap B_\alpha = \emptyset$

$\underline{\alpha = 0}$ $\quad A_0 = B_0 = \emptyset$

$\underline{\alpha = \beta + 1}$ Suppose, we have A_β, B_β.

Consider $T_\beta \in \mathrm{STree}$

$$|[T_\beta]| = 2^{\aleph_0}$$

By (*), $|A_\beta| = |\beta| < 2^{\aleph_0}$

$$|B_\beta| = |\beta| < 2^{\aleph_0}$$

$$\Longrightarrow |A_\beta \cup B_\beta| < 2^{\aleph_0}.$$

$$\Longrightarrow [T_\beta] \setminus (A_\beta \cup B_\beta) \neq \emptyset$$

[even better: it has $2^{\aleph_0}$ many elements]

Define
$$a_\beta := c\left([T_\beta] \setminus (A_\beta \cup B_\beta)\right).$$

$$b_\beta := c\left(\underbrace{[T_\beta] \setminus (A_\beta \cup B_\beta \cup \{a_\beta\})}\right)$$

This is still non-empty since
$|[T_\beta] \setminus (A_\beta \cup B_\beta)| = 2^{\aleph_0}$.

$$A_\alpha := A_\beta \cup \{a_\beta\}$$
$$B_\alpha := B_\beta \cup \{b_\beta\}$$
$$|A_\alpha| = |\beta+1| = |\alpha| = |B_\alpha|.$$

[So, we checked IH (*).]

<u>α is limit ordinal</u> For all $\beta < \alpha$ A_β, B_β are defined & satisfy (*).

$$A_\alpha := \bigcup_{\beta<\alpha} A_\beta$$
$$B_\alpha := \bigcup_{\beta<\alpha} B_\beta$$

Obviously, $|A_\alpha| = |\alpha| = |B_\alpha|$. So (*) is satisfied.

$$A := \bigcup_{\alpha < 2^{\aleph_0}} A_\alpha$$

$$B := \bigcup_{\alpha < 2^{\aleph_0}} B_\alpha$$

<u>Note 1</u> $A \cap B = \emptyset$

[If not: $a_\alpha = b_\beta$ for some α, β.
w.l.o.g. $\alpha \leq \beta$; this contradicts the choice of b_β.]

<u>Note 2</u> $|A| = 2^{\aleph_0} = |B|$

[good: since this was a necessary condition for A and B being non-determined.]

<u>Claim</u> A is not determined.
[B isn't either].

<u>Proof of claim.</u> If it is, then there is $T \in$ STrees, so say $\alpha < 2^{\aleph_0}$ with $T = T_\alpha$ s.t.

either $[T_\alpha] \subseteq A$ CONTRADICTION

or $[T_\alpha] \cap A = \emptyset$ CONTRADICTION

Consider $a_\alpha, b_\alpha \in [T_\alpha]$

$\boxed{a_\alpha \in A}$, $b_\alpha \in B$

$\Rightarrow \boxed{b_\alpha \notin A}$

This finishes the proof. q.e.d.

DISCUSSION

We used AC to produce a non-determined set.

[Usually: AC implies existence of pathologies, e.g., a non-Lebesgue measurable set, or Banach-Tarski decomposition of the unit ball.]

AC does not produce a constructive method for the pathological objects, since the construction depends on the choice fn.

MOTTO [Hope] If a set A is "NICE" or "SIMPLE" then it's not pathological.

Goal Make "nice" and "simple" precise and prove that nice sets are determined.