INFINITE GAMES

LECTURE III
27 Jan 2021

M move set

$M^{<\omega}$ positions

M^{ω} runs

$A \subseteq M^{\omega}$ payoff sets

$G(A)$ game with payoff set A

T tree on M

$T \subseteq M^{<\omega}$

$[T]$ branches through T

$[T] \subseteq M^{\omega}$

$\sigma : M^{<\omega} \longrightarrow M$ strategy

$\sigma * \tau \in M^{\omega}$ $\longrightarrow$ winning strategy for $\underline{\mathrm{I}}$ / $\underline{\mathrm{II}}$.

<u>Definition</u> Let σ be a strategy. We define the I-strategic tree and the II-strategic tree on M as follows:

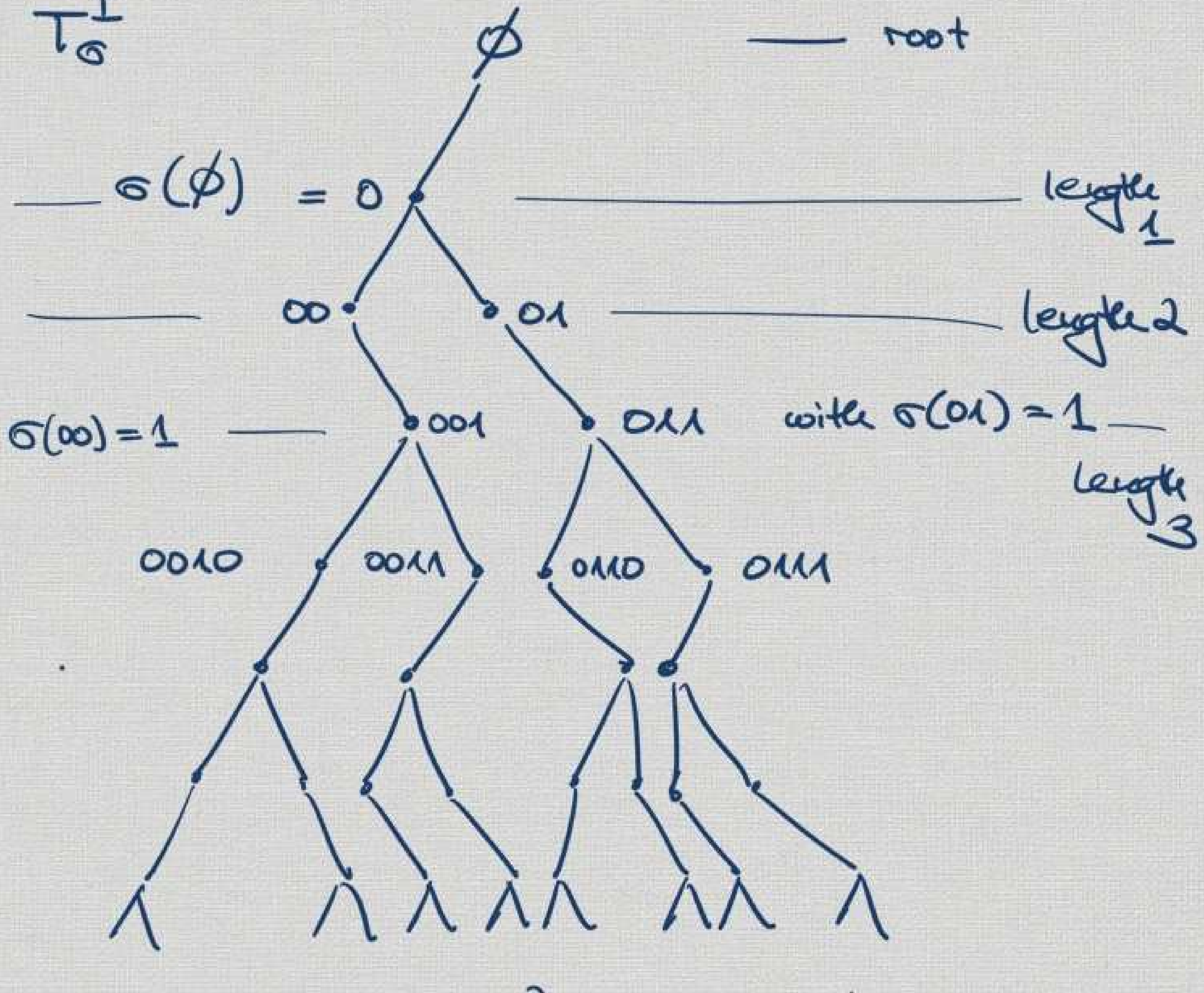
$$T_\sigma^{\mathrm{I}} := \{ s \in M^{<\omega} \,;\, \forall n \ s(2n) = \sigma(s \restriction 2n) \}$$

$$T_\sigma^{\mathrm{II}} := \{ s \in M^{<\omega} \,;\, \forall n \ s(2n+1) = \sigma(s \restriction 2n+1) \}$$

T is called strategic if there is σ s.t. $T = T_\sigma^{\mathrm{I}}$ or $T = T_\sigma^{\mathrm{II}}$.

Draw such a tree. $M = \{0,1\}$

T_σ^I



etc.

$\underline{\mathrm{II}}$-strategic trees look the same except that we have branching in odd length nodes and no branching in even length nodes.

Observe

$$T_\sigma^{I} = \{ (\sigma * \tau) \restriction n \; ; \; \tau \text{ any strategy } \& \; n \in \mathbb{N} \}$$

$$T_\sigma^{II} = \{ (\tau * \sigma) \restriction n \; ; \; \tau \text{ any strategy } \& \; n \in \mathbb{N} \}$$

Therefore

$$[T_\sigma^{I}] = \{ \sigma * \tau \; ; \; \tau \text{ any strategy} \}$$

$$[T_\sigma^{II}] = \{ \tau * \sigma \; ; \; \tau \text{ any strategy} \}$$

PROPOSITION

(1) σ is a w.s. for I in $G(A)$

$$\Longleftrightarrow$$

$$[T_\sigma^{I}] \subseteq A .$$

(2) σ is a w.s. for II in $G(A)$

$$\Longleftrightarrow$$

$$[T_\sigma^{II}] \cap A = \emptyset$$

$$\Longleftrightarrow$$

$$[T_\sigma^{II}] \subseteq M^\omega \setminus A .$$

Also: A is determined iff
either A contains $[T^{\mathrm{I}}_{\sigma}]$ for some σ
or $M^{\omega} \setminus A$ contains $[T^{\mathrm{II}}_{\sigma}]$ for some σ.

NOTATION If $s, t \in M^{<\omega}$, we write st for the concatenation of s & t. If $x \in M^{\omega}$, $s \in M^{<\omega}$, then similarly $sx \in M^{\omega}$ is the concatenation.

If t is a length 1 sequence, say, $t = \langle m \rangle$, we also write sm for $st = s\langle m \rangle$.

We write $lh(s) = dom(s)$ for the length of s.

Definition ① If T is a tree and $s \in T$ we say s is a splitting node in T if there are $m \neq m'$ s.t. both

$$sm, sm' \in T$$

② T is perfect if for each $s \in T$ there is a $t \supseteq s$ s.t. $t \in T$ and t is splitting in T.

[Remark: Every strategic tree is perfect.]

(3) $A \subseteq M^\omega$ is perfect if there is a perfect tree T s.t.

$$A = [T].$$

Remark. Compare to the topological notion of a perfect set: closed without isolated points. We'll see later (~ Lecture VI ?) that these two notions are the same.

Theorem (Cantor). Suppose $A \subseteq 2^\omega = \{0,1\}^\omega$ is perfect and non-empty. Then A has cardinality $2^{\aleph_0}$.

Proof. $A \subseteq 2^\omega$ & $|2^\omega| = 2^{\aleph_0}$.

$$\Longrightarrow |A| \leq 2^{\aleph_0}.$$

So, by Cantor-Schröder-Bernstein, it's enough to show that there is an injection from 2^ω into A.

We define this via a function

$$\varphi : 2^{<\omega} \longrightarrow T$$

where T is perfect s.t. $A = [T]$. Define φ by recursion: (φ is called a Cantor scheme)

$$\varphi(\emptyset) := \emptyset$$

Suppose $\varphi(s) = t \in T$.

If need take the minimal one.

Since T was perfect, find $u \supseteq t$, $u \in T$ that is splitting: $u0, u1 \in T$.

$\varphi(s0) := u0$

$\varphi(s1) := u1$.

This finishes the def'n of φ.

(*) $\hat{\varphi} : 2^{\omega} \longrightarrow [T] = A$

by $\hat{\varphi}(x) := \bigcup_{n \in \mathbb{N}} \underline{\varphi(x \restriction n)}$

We need to check some things:

1. $\mathrm{lh}(\varphi(x \restriction n)) \geq n$.
2. $\varphi(x \restriction n) \subseteq \varphi(x \restriction m)$ if $n \leq m$

$\Longrightarrow \hat{\varphi} : 2^{\omega} \longrightarrow 2^{\omega}$

3. $\hat{\varphi}(x) \restriction n \subseteq \varphi(x \restriction k)$ for some k

so $\hat{\varphi}(x) \restriction n \in T$, so $\hat{\varphi}(x) \in [T]$

$\Longrightarrow$ (*)

Remains to show that $\hat{\varphi}$ is an injection:

$x \neq y$: find n s.t.

$$x \restriction n = y \restriction n \text{, but}$$

$$\underset{\text{WLOG}}{x(n)} \neq y(n)$$

$$x(n) = 0, \quad y(n) = 1$$

$$\underset{\text{ends in } 0}{\varphi(x \restriction n+1)} \neq \underset{\text{ends in } 1}{\varphi(y \restriction n+1)}$$

$$\longrightarrow \underbrace{\bigcup_{k \in \mathbb{N}} \varphi(x \restriction k)}_{= \hat{\varphi}(x)} \neq \underbrace{\bigcup_{k \in \mathbb{N}} \varphi(y \restriction k)}_{= \hat{\varphi}(y)}.$$

q.e.d.

__Remark__ If $|M| \geq 2$ and T is a perfect tree on M, then the same proof shows

$$2^{\aleph_0} \leq |[T]|.$$

Corollary If $|M| \geqslant 2$, then

(a) if player I has a w.s. in $G(A)$, then $|A| \geqslant 2^{\aleph_0}$.

(b) if player II has a w.s. in $G(A)$ then $|M^{\omega} \setminus A| \geqslant 2^{\aleph_0}$.

[This follows from:
1. strategic trees are perfect
2. perfect sets are large
3. w.s. means "includes str. tree".]

- Note that if $A \subseteq M^{\omega}$ with $|M| \geqslant 2$, then (either) $|A| \geqslant 2^{\aleph_0}$ or $|M \setminus A| \geqslant 2^{\aleph_0}$.
- Corollary gives necessary condition on when a fixed pl. has a w.s., but no non-trivial necessary condition for determinacy.

SUFFICIENT CONDITIONS

Let's do the following as a warm-up:

Prove that if A is countable, then player II has a w.s. in $G(A)$.

<u>Proposition</u> If $A = \{a_i ; i \in \mathbb{N}\}$ is countable, then player $\underline{\mathrm{II}}$ has a w.s. in $G(A)$.

<u>Proof.</u> In $\underline{\mathrm{II}}$'s round k [that means: digit $2k+1$], $\underline{\mathrm{II}}$ takes care of a_k.

Play $1 - a_k(2k+1)$. [assume again we're playing on $M = \{0,1\}$]

Strategy τ is

IGNORE EVERYTHING
PLAYER $\underline{\mathrm{I}}$ DOES AND
BLINDLY PLAY $1 - a_k(2k+1)$
IN YOUR k-th MOVE.

$$\begin{aligned}(\sigma * \tau)_{\underline{\mathrm{II}}}(k) &= (\sigma * \tau)(2k+1) \\ &= 1 - a_k(2k+1) \\ &\neq a_k(2k+1)\end{aligned}$$

So $\sigma * \tau \neq a_k$ for arbitrary k, so $\sigma * \tau \notin A$. Thus τ is winning. q.e.d.