INFINITE GAMES

M set of moves
[In most situations: \( M = \omega \).

\[ M = \omega \]
\[ n \in \omega \]
\[ n \leq \omega \]
\[ n = \{ 0, 1, \ldots, n - 1 \} \]

Functions are sets of ordered pairs:
\[ M^n := \{ (s, j) : s : n \rightarrow M \} \]

If \( s \in M^n \) and \( t \in M^k \) \( k > n \)
then \( s \subseteq t \) is the same as
\( s \) is an initial segment of \( t \)
\( t \) is an extension of \( s \)

We can write if \( n < m \) and
\( s \in M^n \), \( s \subseteq s \in M^m \).

\[ M^{< \omega} := \bigcup_{n \in \omega} M^n \]

The set of all finite sequences of elements of \( M \)
These will be called the positions
\[ M^{\omega} := \{ (x, j) : x : N \rightarrow M \} \]
These are called runs or plays.
If \( x \in M^n \) is a run and \( x \in M \), then

\[
x_{\lfloor n \rfloor} : n \rightarrow M
\]

is the position that the play producing \( x \) was in after \( n \) rounds.

### The games on \( M \)

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<th>( m_0 )</th>
<th>( m_2 )</th>
<th>( m_4 )</th>
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<td>II</td>
<td>( m_1 )</td>
<td>( m_3 )</td>
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We are restricting our attention to games where I, II play in alternation and player I starts.

[Remark: More general games can be described by these; see later.]

Then \( x(i) : = m_i \) is the run produced by the game and

\[
s := x_{\lfloor n \rfloor} \text{ is the n-th position.}
\]
If $x \in M^\omega$, we write

$$x_I(i) := x(2i)$$

$$x_{II}(i) := x(2i+1)$$

$x_I, x_{II} \in M^\omega$ correspond to the moves made by pla. I, II, respectively.

If $x, y \in M^\omega$, we write $x \leftrightarrow y$ [INTERLEAVING]

for the seq. $z$ defined by

$$z(u) := \begin{cases} x(k) & \text{if } u = 2k \\ y(k) & \text{if } u = 2k+1 \end{cases}$$

Clearly, $x_I \leftrightarrow x_{II} = x$.

If $A \subseteq M^\omega$, we call $A$ a payoff set. In the game $G(A)$ we say that player I wins if $x \in A$; otherwise, player II wins.
We call any function 
\( \sigma : M \leftarrow \omega \rightarrow M \)
a strategy.

If \( \sigma, \tau \) are strategies, we can play them against each other:

\[ \sigma \ast \tau \]

\( \sigma \ast \tau \) is defined by

\[ (\sigma \ast \tau)(2n) := \sigma((\sigma \ast \tau)(2n)) \]
\[ (\sigma \ast \tau)(2n+1) := \tau((\sigma \ast \tau)(2n+1)) \]

We say that \( \sigma \) is **WINNING FOR I IN G6A** if

\[ \forall A : \sigma \ast \tau A. \]
We say that $T$ is **WINNING** in $(\mathcal{A})$ if

$T \not\models \varphi \land \psi$.

We say that a set $\mathcal{A}$ is **determined** if one of the two players has a winning strategy in $(\mathcal{A})$.

Remark: Clearly, at most one can have a w.s. [otherwise, play them against each other.]

* It is not obvious (and not true) that every set is determined.

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A few remarks on the generality of our notions:

A set $T \subseteq M^{<\omega}$ is called a tree if it is closed under initial segments, i.e., if $\text{set} \in T$ and $t \in T$, then $t \subseteq T$.

* We shall see that $AC \Rightarrow$ there are non-determined sets but "every set is determined" (Axiom of Determinacy) is consistent with ZF.
In this set-theoretic notion of tree, each node in the tree (= positive) has in it all of the information about the path from the root \( \emptyset \) to it.

If \( T \) is a tree and \( x \in M^{\omega} \), we say that \( x \) is a **branch through** \( T \) if for all \( n \in \mathbb{N} \), \( x \upharpoonright n \in T \).

We write \( [T] \) for the set of branches through \( T \) (sometimes: the body of \( T \)).

Example: \( M^{<\omega} \) is a tree.

\[ [M^{<\omega}] = M^{\omega} \]
We can think of a tree $T$ as "finitary" rules for a game:

If $x \notin [T]$, then there is an $n$ s.t. $x \notin T$.

Find $n$ minimal with this property:

If $n$ is odd, then player I left the tree.

If $n$ is even, then player II left the tree.

Define a game $G(A, T)$ where $A \subseteq [T]$ and $T$ is a tree on $N$.

I

II

If $x \in A$, then player I wins.

If $x \notin [T]$ and the least $n$ s.t. $x \notin T$ is even, then player I wins.

In all other case, player II wins.
This is not any more general chase the \( G(A) \) game.

\[
A_T := \left\{ x \in M^0 : x \in A \iff x \notin \{T \} \right\}
\]

and the least \( n \) s.t. \( x \in M \neq T \) is even.

These \( G(A; T) \) and \( G(A_T) \) are the same game.

This idea gives us a lot of flexibility with the move set.  

**Ex 1** Suppose moves for \( I \) are \( X \) and \( Y \).

Form \( M := X \cup Y \)

\( A \subseteq M^0 \)

\( T := \{ s(j) \mid j \text{ if } n \text{ is even } \} \cup \{ s(w) \mid j \text{ if } n \text{ is odd } \} \)

These \( G(A; T) \)

**Ex 2** Suppose \( I \) can always make two moves, but \( II \) can only make one.

Form \( M := X^2 \cup X^1 \)

[Apply idea of Ex 1 with  \( X = X^2 \) and \( Y = X^1 \).]
Ex3 If \( X \subseteq Y \), then every game \( G(A) \) on \( X \) can be thought of as a game on \( Y \):

\[
G(A; T) \quad \text{where} \quad T := X^{<\omega} \subseteq Y^{<\omega}.
\]