INFINITE GAMES Lecture II

25 Jan 2021

M set of moves

[In most situations: $M = \mathbb{N}$.]

SET THEORETIC LINGO

$\mathbb{N} = \omega$

$n \in \mathbb{N}$ $\quad n = \underline{\{0, 1, \ldots, n-1\}}$

Functions are sets of ordered pairs:

$$M^n := \{ s \,;\, s : \underline{\underline{n}} \longrightarrow M \}$$

If $s \in M^n$ and $t \in M^k$ $\quad k > n$

then $\underline{s} \subseteq \underline{t}$. is the same as

"s is an initial segment of t"

"t is an extension of s"

We can write if $m < n$ and

$s \in M^n$; $\underline{s \upharpoonright m \in M^m}$.

$$M^{<\omega} := \bigcup_{n \in \mathbb{N}} M^n$$

the set of all finite sequences of elements of M

These will be called the POSITIONS.

$$M^\omega := \{ x \,;\, x : \mathbb{N} \longrightarrow M \}$$

These are called RUNS or PLAYS.

If $x \in M^\omega$ is a run and $n \in \mathbb{N}$, then

$$x \upharpoonright n : n \longrightarrow M$$

is the position that the play producing x was in after n rounds.

The games on M

I	m_0	m_2	m_4	...
II	m_1	m_3	m_5	...

We are restricting our attention to games where I, II play in alternation and player I starts.

[Remark: More general games can be described by these; see later.]

Then $x(i) := m_i$ is the run produced by the game and $s := x \upharpoonright n$ is the n-th position.

If $x \in M^\omega$, we write

$$x_{\mathrm{I}}(i) := x(2i)$$

$$x_{\mathrm{II}}(i) := x(2i+1)$$

$x_{\mathrm{I}}, x_{\mathrm{II}} \in M^\omega$ correspond to the moves made by pl. I, II, respectively.

LENGTH ω

If $x, y \in M^\omega$, we write $x * y$ [INTERLEAVING]

for the seq. z defined by

$$z(n) := \begin{cases} x(k) & \text{if } n = 2k \\ y(k) & \text{if } n = 2k+1 \end{cases}$$

Clearly, $x_{\mathrm{I}} * x_{\mathrm{II}} = x$.

If $A \subseteq M^\omega$, we call A a **payoff set**. In the game $G(A)$ we say that player I wins a run $x \in M^\omega$ if $x \in A$; o/w, player II wins.

WIN-LOSE

We call any function

$$\sigma : M^{<\omega} \longrightarrow M$$

a strategy.

If σ, τ are strategies, we can play them against each other:

$\boxed{\sigma * \tau}$

$\sigma * \tau \in M^\omega$ defined by

[Note that each strategy in this sense can be thought of as a strategy for $\underline{I}$ PLUS strategy for $\underline{II}$:

$$O := \bigcup_{n \text{ is odd}} M^n$$

$$E := \bigcup_{n \text{ is even}} M^n .$$

$\sigma \upharpoonright E$ is a str. for $\underline{I}$
$\sigma \upharpoonright O$ is a str. for $\underline{II}$

SO: there is redundance in our notion.]

$$(\sigma * \tau)(2n) := \sigma((\sigma * \tau) \upharpoonright 2n)$$
$$(\sigma * \tau)(2n+1) := \tau((\sigma * \tau) \upharpoonright 2n+1)$$

We say that σ is WINNING FOR I IN $G(A)$ if $\forall \tau$ $\sigma * \tau \in A$.

We say that τ is <u>WINNING FOR II IN $G(A)$</u> if

$$\forall \sigma \quad \sigma * \tau \notin A.$$

We say that a set A is <u>DETERMINED</u> if one of the two players has a winning strategy in $G(A)$.

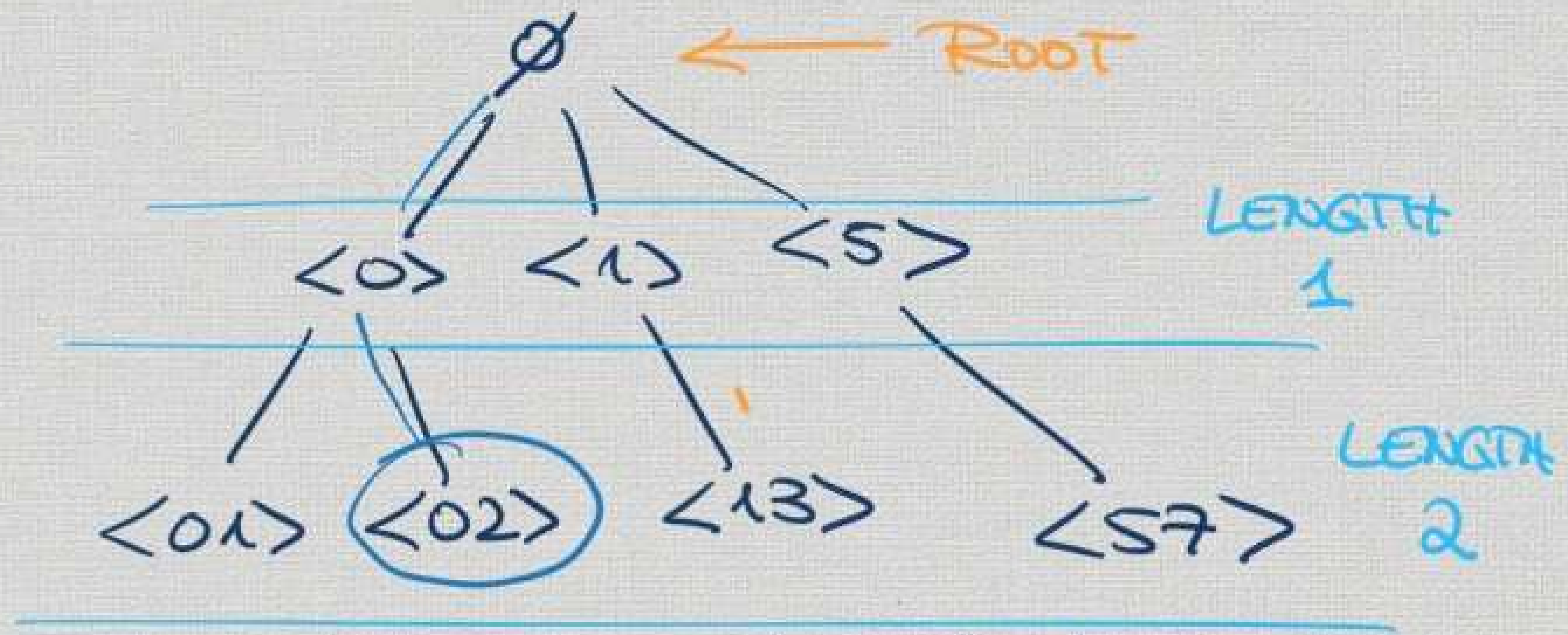
<u>Remark</u> • Clearly, at most one can have a w.s. [otherwise, play them against each other.]

• It is not obvious (and not true*) that every set is determined.

A few remarks on the generality of our notions:

A set $T \subseteq M^{<\omega}$ is called a <u>tree</u> if it is closed under initial segments, i.e., if $s \in T$ and $t \subseteq s$, then $t \in T$.

* We shall see that AC $\Longrightarrow$ there are non-determined sets; but "every set is determined" (Axiom of Determinacy) is consistent with ZF.



In this set-theoretic notion of tree, each node in the tree (= position) has in it all of the information about the path from the root $\emptyset$ to it.

If T is a tree on M and $x \in M^\omega$, we say that x is a **branch through T** if for all $n \in \mathbb{N}$, $x \upharpoonright n \in T$.

We write $[T]$ for the set of branches through T (sometimes: the body of T)

Example $M^{<\omega}$ is a tree

$$[M^{<\omega}] = M^\omega .$$

We can think of a tree T as "finitary" rules for a game:

if $x \notin [T]$, then there is an n s.t. $x \upharpoonright n \notin T$.

Find n minimal with this property:

if n is odd, then player I left the tree;

if n is even, then player II left the tree.

Define a game $G(A;T)$ where $A \subseteq [T]$ and T is a tree on M.

I	m_0		m_2		$\cdots$
II		m_1		m_3	$\cdots$

produce $x \in M^\omega$

If $x \in A$, then player I wins.

If $x \notin [T]$ and the least n s.t. $x \upharpoonright n \notin T$ is even, then player I wins.

In all other case, player II wins.

This is not any more general than the $G(A)$ game.

$$A_T := \{ x \in M^\omega ; \ x \in A \text{ or } x \notin [T] \text{ and the least } n \text{ s.t. } x \upharpoonright n \notin T \text{ is even} \}$$

Then $G(A;T)$ and $G(A_T)$ are the same game.

This idea gives us a lot of flexibility with the move set:

<u>Ex 1</u> Suppose moves for I are in X
" " II are in Y

Form $M := X \cup Y$

$A \subseteq M^\omega$

$$T := \left\{ s ; \begin{array}{l} \text{if } n \text{ is even } s(n) \in X \\ \text{if } n \text{ is odd } s(n) \in Y \end{array} \right\}$$

Then $G(A;T)$

<u>Ex 2</u> Suppose I can always make two moves in X, but II can only make one.

Form $M := X^2 \cup X^1$

[Apply idea of Ex 1 with $X = X^2$
$Y = X^1$.]

<u>Ex 3</u> If $X \subseteq Y$,
then every game $G(A)$ on X can be thought of as a game on Y:

$$G(A; T)$$

where $T := X^{<\omega} \subseteq Y^{<\omega}$.