

INFINITE GAMES

PART III
LENT TERM 2021

LECTURE I

[22 JANUARY 2021]

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Schedule: Lectures: Mondays, Wednesdays, Fridays, 11:12-12:00 ET.

Example Classes:

- #1: Monday 11 January 2021, 2:30-3:45PM: Infinite Games
- #2: Monday 18 January 2021, 2:30-3:45PM: Games
- #3: Monday 25 January 2021, 2:30-3:45PM: Games
- #4: Friday 15 May 2021, 1:30-2:45PM: Games

Revision Session: Friday, 27 May 2021, 1:30-2:45PM

Infinite Games
Part III of the Mathematical Topics
Lent Term 2021

Friday, 15 January 2021: First Lecture: Well-ordered games; perfect information and perfect recall games. Discussion of the types of games that will not be covered in the lecture series—more than two players, cooperative games, imperfect information, & local (partial) memory. Games (1971), see Rubin's article and his lecture from 1987. Mancosu (1992); Gale & Stromquist (1982); Stromquist and the California rockshelters. Mathematics of Determinacy, Games, positions, values, strategies, the result of two players playing against each other. This notion is a running theme.

Monday, 18 January 2021: Second Lecture: Countable Games. Determinacy. Perfect games. Examples. Proof of determinacy of positional games with finite winning trees. Strategic analysis. Examples of games with winning strategy. Does every game have a winning strategy? Open problems concerning Ramsey games.

Wednesday, 20 January 2021: Third Lecture: Iteration and infinite sequences. Games, iteration of games, concatenation, the type of a sequence and the type of a sequence. Strategic trees and a formulation of having a winning strategy in terms of strategic trees. Necessary condition for being a well-ordered game. Can player II win some of the components of the game?

Friday, 22 January 2021: Fourth Lecture: Multiplayer games. Iterated games. Iterated games, iterations and concatenations in strategy. Perfect games. Zermelo's theorem. Recurrence induction and construction in strategy. Perfect games. The Gale-Stewart theorem. Perfect games are determined. First half of the proof: transfinite recursive induction of the player winning.

Monday, 25 January 2021: Fifth Lecture: Well-founded trees and their heights formula. Continuation of the Gale-Stewart proof that the transfinite recursion terminates. Proof that the resulting game winning strategies. The result of this implies the existence of non-determined sets.

Wednesday, 27 January 2021: Sixth Lecture: Game regularity and its topology, dimension, metric dimension, metric, The Gale-Glenn theorem in its topological formulation. All open sets are determined. Relative understanding of convergence. The representation of closed sets. Continuity of the set of open sets (closed sets). The Borel hierarchy. The Borel hierarchy.

Friday, 29 January 2021: Seventh Lecture: The Borel hierarchy is a sigma-finite ordered wellfounded hierarchy. On measure-theoretic games. If tree height is countable or generic, it has height at most \aleph_1 . A Σ_1^1 set can be determined in general. Σ_1^1 sets are not always nonmeasurable (cf. Lowe & Reinhardt 2007). A brief overview of determinacy in the Borel hierarchy (without proofs). Σ_1^1 determinacy (Miller 1985), Σ_1^1 determinacy (Gale 1962), Σ_1^1 determinacy (Prikry 1976), Borel determinacy (Mazur 1970), Universal sets.

Monday, 1 February 2021: Eighth Lecture: Games on wellfounded sets. The extent of countable wellfoundedness. A long (Connes 1973), short (2000) proof.

Wednesday, 3 February 2021: Ninth Lecture: Projective wellfoundedness, closure properties. Variation on the use of the Power of Choice: variation of Σ_1^1 under recursive ultrafilter-semidecidable filters. In the Prikry model, every set is Σ_1^1 (universal sets). The universal set theorem: The Universal Set Theorem: first use of the power (construction of a universal open set).

Friday, 5 February 2021: Tenth Lecture: Proof of the Lévy-Solovay Set Theorem: dividing coding infinite sequences of elements of Borel spaces in an environment of Borel spaces. Lévy-Solovay's idea: the class of the Borel sets are closed under continuous mappings. Solovay's counterexample (without proof): recursive definition of the projective hierarchy. In the early 1970s, it was known that determinacy in the projective hierarchy needed to rely on forcing axioms of different strength (as proved in DPC 1970).

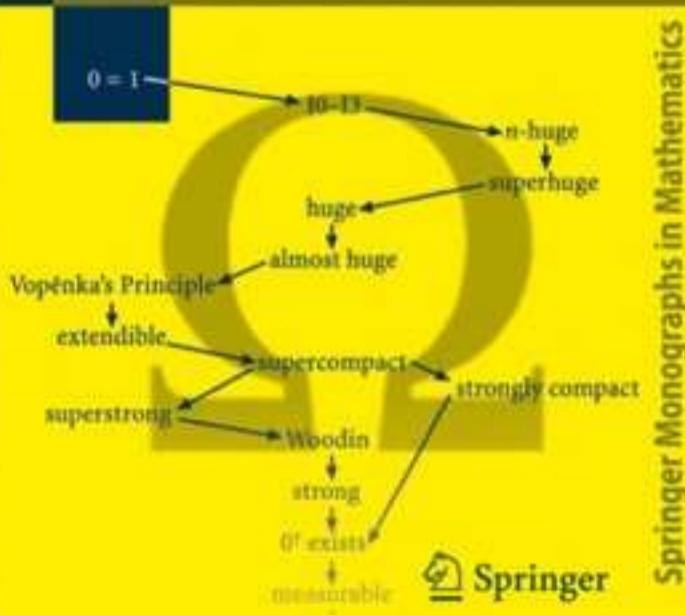
Monday, 8 February 2021: Eleventh Lecture: Proof of Solovay's theorem (continued). Recursive properties. Lévy-Solovay's theorem: the Borel property, Share Category Theorem (continued), the projective hierarchy does not collapse (end of proof).

Monday, 15 February 2021: Twelfth Lecture: Proof of Solovay's theorem (continued). Recursive properties. Lévy-Solovay's theorem: the Borel property, Share Category Theorem (continued), the projective hierarchy does not collapse (end of proof).

AKIHIRO KANAMORI

The Higher Infinite

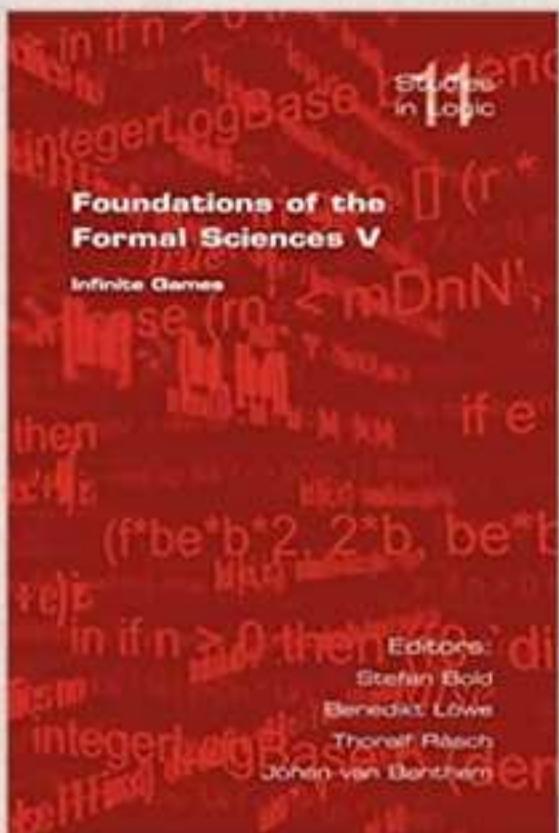
Second Edition



Chapter 6 "Determinacy"

- § 27 "Infinite Games"
§ 28 "AD and Combinatorics"

Topic of the course:



TWO-PLAYER
LENGTH ω
WIN-LOSE
PERFECT INFORMATION
PERFECT RECALL

} GAMES

1913

ÜBER EINE ANWENDUNG DER MENGENLEHRE AUF
DIE THEORIE DES SCHACHSPIELS

VON E. ZERMEO.

Die folgenden Betrachtungen sind unabhängig von den besonderen Regeln des Schachspiels und gelten prinzipiell ebenso gut für alle ähnlichen Verstandesspiele, in denen zwei Gegener unter Ausschluss des Zufalls gegeneinander spielen; es soll aber der Bestimmtheit wegen hier jeweils auf das Schach als das bekannteste aller derartigen Spiele exemplifiziert werden. Auch handelt es sich nicht um irgend eine Methode des praktischen Spiels, sondern lediglich um die Beantwortung der Frage: kann der Wert einer beliebigen während des Spiels möglichen Position für eins der mathematisch-objektiv bestimmten mehr subjektiv-psychologischen in Bezug genommen zu werden sondernen Fällen möglich ist, insbesondere von Positionen, in denen ein Anzahl von Zügen das Matt der Position auch in anderen in der unübersehbaren Komplikationen Hindernis findet, in dem Sinn hat, scheint mir dochstellung dürfte für die praktische wir sie in den Lehrbüchern des Se im folgenden zur Lösung des " und dem "logischen Kalkül " mathematischen Disziplinen in le Gesamtheiten handelt.



Ernst Zermelo

German logician

Ernst Friedrich Ferdinand Zermelo was a German logician and mathematician, whose work has major implications for the foundations of mathematics. He is known for his role in developing Zermelo–Fraenkel axiomatic set theory and his proof of the well-ordering theorem. [Wikipedia](#)

Born: July 27, 1871, Berlin

Died: May 21, 1953, Freiburg im Breisgau

Zermelo's
Theorem

Every finite game is determined.

① Two players

We have two players : I and II.

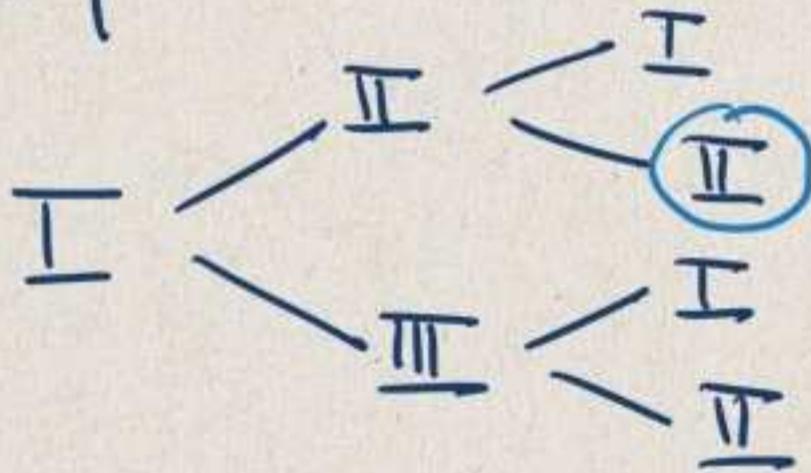
Three-player games do not in general admit w.s. :

I, II, III Player I is given a gold coin.

ROUND 1 I can give it to II or to III.

ROUND 2 Whoever has the coin can give it to I or to II.

The person with the coin wins.



No one has a winning strategy!

However COALITIONS

{I, II}

can

{I, III}

ensure

{II, III}

a win.

→ COOPERATION!

② LENGTH ω

In game theory in economics, there is research on potentially infinite games.

PRISONER'S DILEMMA

2,2	1,3
3,1	0,0

Think of it as a repeated game.

→ ASYMPTOTIC BEHAVIOUR

→ EVOLUTIONARY PHENOMENA

If you think of truly infinite games, new changes.

EXAMPLE

PFG PRIME FACTOR GAME

I $k_0 \quad k_1 \quad k_2 \quad k_3 \quad \dots \quad K = \{k_i : i \in \mathbb{N}\}$

II $p_0 \quad p_1 \quad p_2 \quad p_3 \quad \dots \quad P = \{p_i : i \in \mathbb{N}\}$

$k_i \geq 2$ natural numbers

p_i are primes

Player II wins if P is the set of all prime factors of K .

Observe :

Player II has a winning strategy

$$k_0 = q_0^{\bar{b}_0} \dots q_m^{\bar{b}_m}$$

Then play

$$\begin{cases} p_0 = q_0 \\ p_1 = q_1 \\ p_2 = q_2 \\ \vdots \\ p_m = q_m \end{cases}$$

$$k_1 = \frac{p_0}{\bar{q}_0} \dots \frac{p_m}{\bar{q}_m}$$

$$p_{m+1} = q_0$$

Clearly, this ensures that
 $\{p_i; i \in \mathbb{N}\}$
 is the set of price factors of K .

If you pause the game at $N \in \mathbb{N}$, then in most runs of the game, the finite sets $\{k_0, \dots, k_N\}$ and $\{p_0, \dots, p_N\}$ do not look like a win for player II.

COMMON OBJECTION:

You can't play these games!

Even though they can't be played, we can prove that a w.s. exists (or doesn't).

Let's modify PFG slightly:

\overline{I}	b_0	b_1	b_2	...
\overline{II}	p_0	p_1	p_2	...

Now player \overline{I} has a w.s.

Take p_0 , find $q \neq p_0$ and play

$$b_i := q^{i+1}.$$

Then $K = \{ q^{i+1}; i \in \mathbb{N} \}$

So \overline{I} wins iff $P = \{q\}$, so \overline{I} loses.

③ WIN-LOSE

ZERO-SUM : fixed pay-off split between the players

Prisoner's Dilemma

2,2	1,3
3,1	0,0

NOT AN EXAMPLE

EXAMPLE :

1,1	0,2
2,0	1,1

WIN-LOSE MEANS: pay-off is completely divisible

In our case, pay-off functions
are characteristic functions of a
payoff set

$$A \subseteq \underline{\text{RUNS}}$$

↑
set of all possible runs

(4) PERFECT INFORMATION

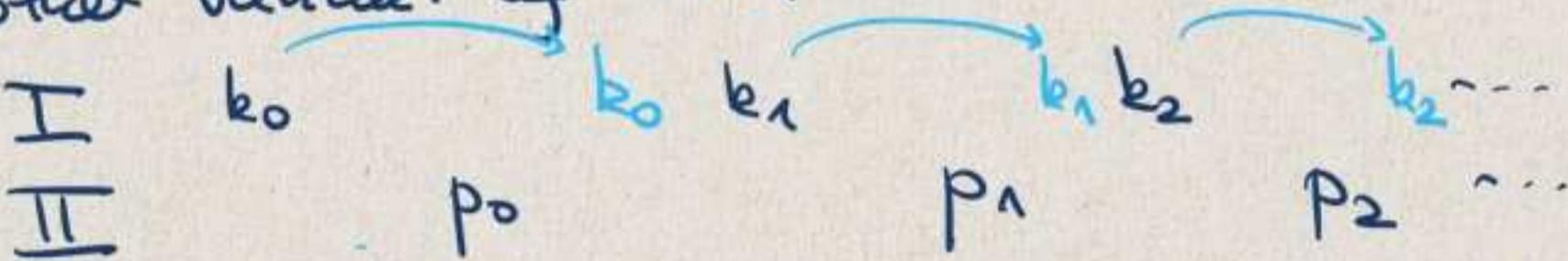
paradigmatic: board games

non-example: card games

↑ [your own hand is only known
to you]

IMPERFECT INFORMATION

Another variant of PFG:



Here, player I picks k_i , but does not have to reveal k_i before player II has played p_i .

Neither of the two players has a w.s. ie this variant.

[The study of these games is closely related to probability.]

(S)

PERFECT RECALL

The opposite would be that players leave a bounded memory.

E.g. PFG + Player II can only remember the last 1000 moves

[This is very relevant in applications of infinite games in computer science.]