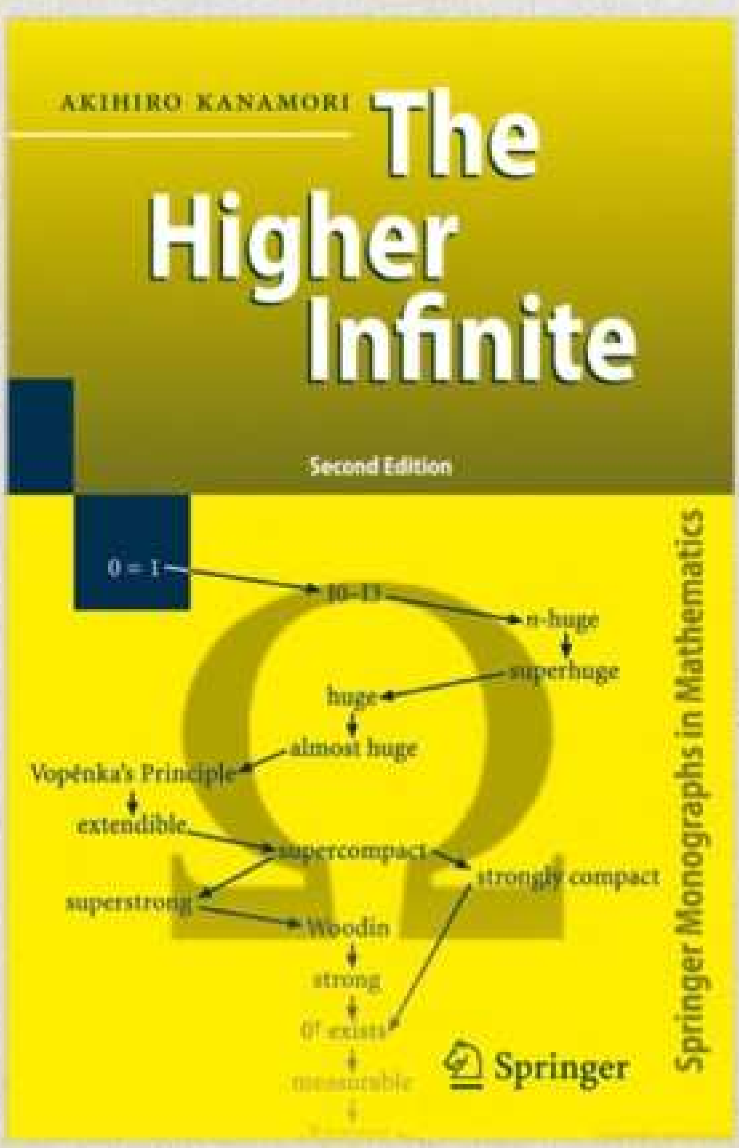
Infinite Games

Part III
Lent Term 2021

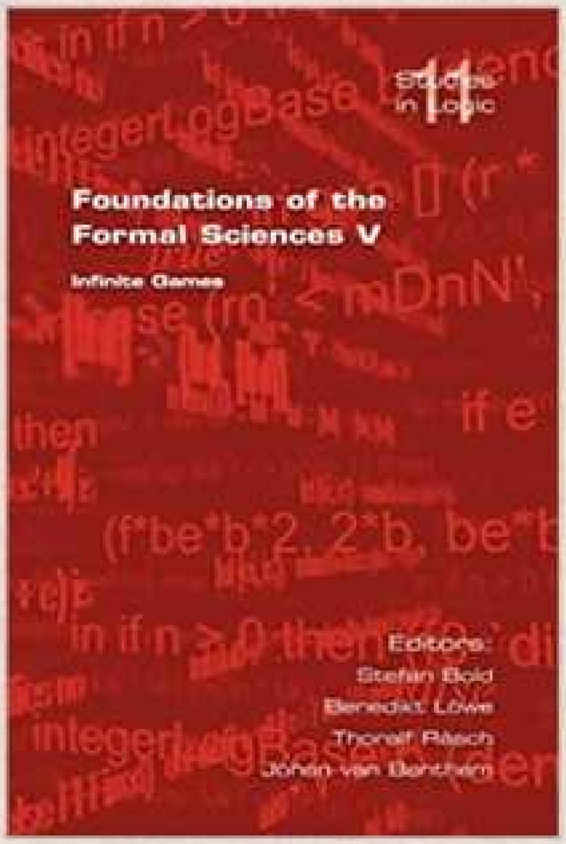
Lecture I [22 January 2021]



Chapter 6
"Determinacy" –

§27 "Infinite Games"
§28 "AD and Combinatorics"

Topic of the course:



TWO-PLAYER
LENGTH ω
WIN-LOSE
PERFECT INFORMATION
PERFECT RECALL

} GAMES

1913

ÜBER EINE ANWENDUNG DER MENGENLEHRE AUF DIE THEORIE DES SCHACHSPIELS

VON E. ZERMELO.

Die folgenden Betrachtungen sind unabhängig von den besonderen Regeln des Schachspiels und gelten prinzipiell ebensogut für alle ähnlichen Verstandesspiele, in denen zwei Gegener unter Ausschluss des Zufalls gegeneinander spielen; es soll aber der Bestimmtheit wegen hier jeweilig auf das Schach als das bekannteste aller derartigen Spiele exemplifiziert werden. Auch handelt es sich nicht um irgend eine Methode des praktischen Spiels, sondern lediglich um die Beantwortung der Frage: kann der Wert einer beliebigen während des Spiels möglichen Position für eine der

athematisch-objektiv bestimmt
mehr subjektiv-psychologischen
n Bezug genommen zu werden
esonderen Fällen möglich ist,
eispiele von Positionen, in denen
en Anzahl von Zügen das Matt
der Position auch in anderen
n der unübersehbaren Kompli-
überwindliches Hindernis findet,
inen Sinn hat, scheint mir doch
tellung dürfte für die praktische
wir sie in den Lehrbüchern des
Sie im folgenden zur Lösung des
" und dem "logischen Kalkül"
mathematischen Disziplinen in
he Gesamtheiten handelt.

Zermelo's Theorem

Every finite such game is determined.

Ernst Zermelo

German logician

Ernst Friedrich Ferdinand Zermelo was a German logician and mathematician, whose work has major implications for the foundations of mathematics. He is known for his role in developing Zermelo–Fraenkel axiomatic set theory and his proof of the well-ordering theorem. Wikipedia

Born: July 27, 1871, Berlin

Died: May 21, 1953, Freiburg im Breisgau

① Two players

We have two players: I and II.

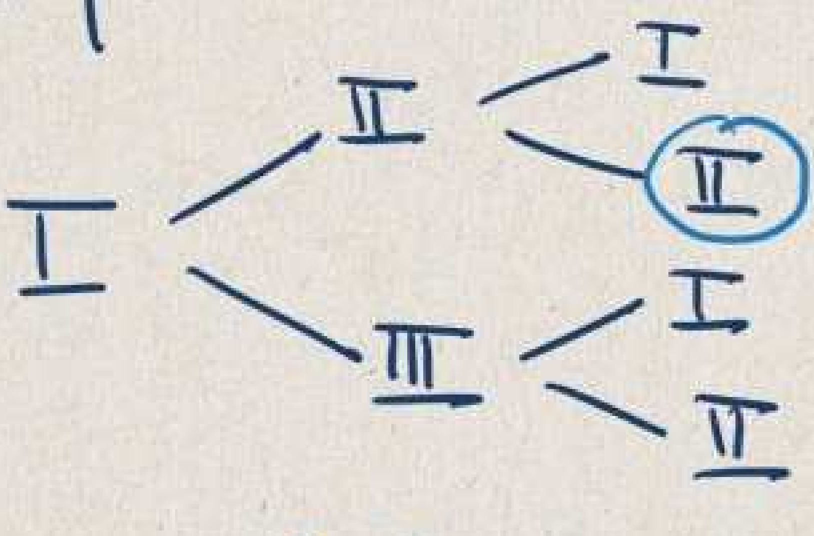
Three-player games do not in general admit w.s.:

I, II, III Player I is given a gold coin.

ROUND 1 I can give it to II or to III.

ROUND 2 Whoever has the coin can give it to I or to II.

The person with the coin wins.



No one has a winning strategy!

However COALITIONS {I, II} can
{I, III} ensure
{II, III} a win.

⟶ COOPERATION!

(2) LENGTH ω

In game theory in economics, there is research on potentially infinite game.

PRISONER'S DILEMMA

2,2	1,3
3,1	0,0

Think of it as a repeated game.

⟶ ASYMPTOTIC BEHAVIOUR

⟶ EVOLUTIONARY PHENOMENA

If you think of truly infinite games, things changes.

EXAMPLE. PFG

PRIME FACTOR GAME

$$\begin{array}{llllll} \mathrm{I} & k_0 & k_1 & k_2 & k_3 & \dots \quad K=\{k_i\}_{i\in\mathbb{N}} \\ \mathrm{II} & p_0 & p_1 & p_2 & p_3 & \dots \quad P=\{p_i\}_{i\in\mathbb{N}} \end{array}$$

$k_i \geq 2$ natural numbers

p_i are primes Player II wins if P is the set of all prime factors in K.

Observe:

Player II has a winning strategy

$$k_0 = q_0^{l_0} \cdots q_m^{l_m}$$

Then play

$$p_0 = q_0$$
$$p_1 = q_1$$
$$p_2 = q_2$$
$$\vdots$$
$$p_m = q_m$$

$$k_1 = \overline{q_0}^{\overline{l_0}} \cdots \overline{q_n}^{\overline{l_n}}$$

$$p_{m+1} = \overline{q_0}$$
$$\vdots$$

Clearly, this ensures that $P = \{p_i ; i \in \mathbb{N}\}$ is the set of prime factors of K.

If you pause the game at $N \in \mathbb{N}$, then in most runs of the game, the finite sets $\{k_0, \dots, k_N\}$ and $\{p_0, \dots, p_N\}$ do not look like a win for player II.

COMMON OBJECTION:

You can't play these games!

Even though they can't be played, we can prove that a w.s. exists (or doesn't).

Let's modify PFG slightly:

I		k_0	k_1	k_2 ...
II	p_0	p_1	p_2	...

Now player I has a w.s.
Take p_0, find $q \neq p_0$ and play

$$k_i := q^{i+1}.$$

Then $K = \{ q^{i+1} ; i \in \mathbb{N} \}$

So II wins iff $P = \{q\}$, so II loses.

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(3) WIN-LOSE

ZERO-SUM : fixed pay-off split between the player

Prisoner's Dilemma

2,2	1,3
3,1	0,0

NOT AN EXAMPLE

EXAMPLE:

1,1	0,2
2,0	1,1

WIN-LOSE MEANS: pay-off is an indivisible $\underline{1}$

In our case, pay-off functions are characteristic functions of a payoff set

$$A \subseteq \underline{\text{RUNs}}$$

set of all possible runs

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④ PERFECT INFORMATION

paradigmatic: board games

non-example: card games [your own hand is only known to you]

IMPERFECT INFORMATION

Another variant of PFG:

$$\begin{array}{lllllll} \text{I} & k_0 & & k_0\ k_1 & & k_1\ k_2 & & k_2 \ldots \\ \text{II} & & p_0 & & p_1 & & p_2 & \ldots \end{array}$$

Here, player I picks k_i, but does not have to reveal k_i before player II has played p_i

Neither of the two players has a w.s. in this variant.

[The study of these games is closely related to probability.]

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(5) PERFECT RECALL

The opposite would be that players have a bounded memory.

E.g. PFG + player II can only remember the last 1000 moves

[This is very relevant in applications of infinite games in computer science.]