Infinite Games
Lent Term 2021
Part III of the Mathematical Tripos
University of Cambridge
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Example Sheet #4

The final Examples Class.
#4: Friday 30 April 2021, 3:30–5pm, Zoom.

Remote student interaction & presentations. As for the first example sheet, we ask you to arrange mathematical discussions about the material of *Infinite Games* in pairs: please arrange virtual meetings with one of the other students taking this course and work on Examples 41 & 49 together, preparing a brief online presentation of these examples for the third Examples Class. Your pair meetings can and should be about both the mathematical content of the two examples and about the practicalities of the presentation.

Marking. On the moodle page, there is an *Assignment* called *Example Sheet #4* in the section *Example Sheets & Classes*. You can submit your work as a single pdf file there. Feel free to submit all of your work; Examples 47 & 50 will be marked. Please submit your work by Thursday noon (i.e., **29 April 2021, 12pm GMT**).

(39) Show in ZF that there is a family of functions $\{g_x\,;\, x \in \omega^\omega\}$ such that for every $x \in \mathrm{WF}$, the function g_x is a surjection from ω^ω into the power set of $\|x\|$.

(40) Work in $\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$. A subset $C \subseteq \omega_1$ is called *closed* if for each $\gamma \in \omega_1$, if $C \cap \gamma$ is unbounded in γ, then $\gamma \in C$; it is called *unbounded* if for each $\alpha \in \omega_1$, there is a $\gamma \in C$ such that $\alpha < \gamma$; it is called *club* if it is closed and unbounded. If $C \subseteq \omega_1$ is uncountable, let $c : \omega_1 \to \omega_1$ be its enumeration function where $c(\xi)$ is the ξth element of C.

  (a) Show that C is club if and only if c is a *normal* function, i.e., increasing (if $\xi < \eta$, then $c(\xi) < c(\eta)$) and continuous at limit ordinals ($c(\lambda) = \bigcup\{c(\xi)\,;\, \xi < \lambda\}$).
  (b) Show that if C is club, then $\{\xi\,;\, c(\xi) = \xi\}$ is club.
  (c) Show that the intersection of countably many club sets is club.
  (d) Show that the diagonal intersection of a family $\{C_\xi\,;\, \xi < \omega_1\}$ of club sets is club.

(41) (Presentation Example). Show in ZFC that the set $\mathcal{C} := \{A \subseteq \aleph_1\,;\,$ there is a club set C with $C \subseteq A\}$ is a non-principal $\aleph_1$-complete normal filter on $\aleph_1$ that is not an ultrafilter. (It is called the *club filter*.) For each of the properties proved, discuss how much choice your proof used. In particular, what part of this still holds in ZF?

(42) We write AD_M for the axiom of determinacy for games with moves in M, i.e., "every $A \subseteq M^\omega$ is determined". As usual, $\mathsf{AD} := \mathsf{AD}_\omega$. Show in ZF that AD_2 is equivalent to AD_ω.

(43) Show in ZF that AD implies that there is no injection from $\aleph_1$ into ω^ω.

(44) Assume ZF+"there is an injection from $\aleph_1$ into ω^ω". Show that ω^ω is a disjoint union of $\aleph_1$ many $\mathbf{\Pi}^0_3$ sets.

(45) Let $\mathcal{D}$ be the collection of determined sets. Show in ZF that AD is equivalent to the statement "$\mathcal{D}$ is a σ-algebra".

(46) Let U be an $\aleph_1$-complete ultrafilter on X and $F : X \to \omega^\omega$. Show that F is constant on a set in U.

(47) (Marked Example). Let $x \in \omega^\omega$. Show that $\mathrm{Cone}(x) := \{y \in \omega^\omega\,;\, x \leq_{\mathrm{D}} y\}$ is unbounded in WF, i.e., for every $\alpha < \omega_1$, there is some $y \in \mathrm{Cone}(x) \cap \mathrm{WF}_{>\alpha}$.

(48) Show that AD implies that there is no injection from $\mathcal{D}_{\mathrm{D}}$ into ω^ω.

(49) (Presentation Example). Work in $\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$. Let $A \subseteq \omega_1$. The following game $\mathrm{G}_{\mathrm{S}*}(A)$ is a variant of the Solovay games of Examples (37) & (38): players I and II play in alternation; player I produces $x \in \omega^\omega$ and player II produces $y \in \omega^\omega$; consider the set $\{(x)_n\,;\, n \in \omega\} \cup \{(y)_n\,;\, n \in \omega\} \subseteq \omega^\omega$; if one of its elements is not in WF, then player II loses if the least m such that $(y)_m \notin \mathrm{WF}$ is smaller than the least m such that $(x)_m \notin \mathrm{WF}$; otherwise player I loses; if all of them are in WF, define

$$\gamma := \sup(\{\|(x)_n\|\,;\, n \in \omega\} \cup \{\|(y)_n\|\,;\, n \in \omega\})$$

and say that player I wins if $\gamma \in A$.

(a) Show that if player I has a winning strategy in $\mathrm{G}_{\mathrm{S}*}(A)$, then A contains a club set (cf. Example (40));

(b) Show that if player II has a winning strategy in $\mathrm{G}_{\mathrm{S}*}(A)$, then the complement of A contains a club set.

Use this to get a different proof of Solovay's Theorem.

(50) (Marked Example). Work in $\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$. Let $A \subseteq \omega_1$ and use the family of surjections $\{g_x\,;\, x \in \omega^\omega\}$ from Example (39). We define the following game $\mathrm{G}_{\mathrm{FM}}(A)$, called the *Friedman-Moschovakis game*: players I and II play in alternation; player I produces $u \in \omega^\omega$ and we split it into $x := u_{\mathrm{I}}$ and $a := u_{\mathrm{II}}$; player II produces $v \in \omega^\omega$ and we split it into $y := v_{\mathrm{I}}$ and $b := v_{\mathrm{II}}$; if $x \notin \mathrm{WF}$, player I loses; if $x \in \mathrm{WF}$, but $y \notin \mathrm{WF}$, player II loses; if both are in WF, consider $X := g_x(a)$ and $Y := g_y(b)$; if $X \neq A \cap \|x\|$, then player I loses; if $X = A \cap \|x\|$, but $Y \neq A \cap \|y\|$, then player II loses; if $X = A \cap \|x\|$ and $Y = A \cap \|y\|$, then player II wins if and only if $\|y\| > \|x\|$.

(a) Show that player I cannot have a winning strategy in $\mathrm{G}_{\mathrm{FM}}(A)$.

(b) Suppose τ is a strategy for player II and there is some $A \subseteq \omega_1$ such that τ is a winning strategy in $\mathrm{G}_{\mathrm{FM}}(A)$. Show that A is uniquely determined.

(c) Prove that AD implies that there is a surjection from ω^ω onto the power set of $\aleph_1$.