Infinite Games
Lent Term 2021
Part III of the Mathematical Tripos
University of Cambridge
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Example Sheet #3

Examples Classes.
#3: Friday 12 March 2021, 3:30–5pm, Zoom.
#4: Friday 30 April 2021, 3:30–5pm, Zoom.

Remote student interaction & presentations. As for the first example sheet, we ask you to arrange mathematical discussions about the material of *Infinite Games* in pairs: please arrange virtual meetings with one of the other students taking this course and work on Examples 29 & 37 together, preparing a brief online presentation of these examples for the third Examples Class. Your pair meetings can and should be about both the mathematical content of the two examples and about the practicalities of the presentation.

Marking. On the moodle page, there is an *Assignment* called *Example Sheet #3* in the section *Example Sheets & Classes*. You can submit your work as a single pdf file there. Feel free to submit all of your work; Examples 27 & 34 will be marked. Please submit your work by Thursday noon (i.e., **11 March 2021, 12pm GMT**).

(26) Let $C \subseteq \omega^\omega \times 2^\omega$ and consider the following *unfolded asymmetric game* $\mathrm{G}^*_{\mathrm{u}}(C)$: player I plays pairs (x_n, s_n) where $x_n \in \omega$ and $s_n \in 2^{<\omega}$ and player II plays $b_n \in 2 = \{0,1\}$. Write $x(n) := x_n$, so $x \in \omega^\omega$ and $y := s_0 b_0 s_1 b_1 \ldots \in 2^\omega$ as in the asymmetric game. Player I wins if $(x,y) \in C$. Let $A := \mathrm{p}C$ and show:

(a) If player I has a winning strategy in $\mathrm{G}^*_{\mathrm{u}}(C)$, then A contains a non-empty perfect subset.

(b) If player II has a winning strategy in $\mathrm{G}^*_{\mathrm{u}}(C)$, then A is countable.

Conclude that $\mathsf{Det}(\mathbf{\Pi}^1_n)$ implies $\mathsf{PSP}(\mathbf{\Sigma}^1_{n+1})$; in particular, $\mathsf{PSP}(\mathbf{\Sigma}^1_1)$ is a theorem of ZFC.

(27) Marked Example. Suppose $M \subseteq V$ are two sets such that $(M, \in) \models \mathsf{ZFC}$, $(V, \in) \models \mathsf{ZFC}$, $\mathrm{Ord} \cap M = \mathrm{Ord} \cap V$, and M is transitive in V, i.e., if $x, y \in V$, $x \in y$, and $y \in M$, then $x \in M$. (This is what we called "M is an inner model of V" in the lectures.) Suppose that $x, y, f \in M$. Show the claims from the lecture:

(a) $M \models$"f is a function from x to y" if and only if $V \models$"f is a function from x to y".

(b) $M \models$"f is a surjective function from x to y" if and only if $V \models$"f is a surjective function from x to y".

(c) If $\alpha, \beta \in V$ are ordinals, then $M \models$"f is a cofinal function from α to β" if and only if $V \models$"f is a cofinal function from α to β".

(d) If $M \models x \in \mathrm{WF}$, then $V \models x \in \mathrm{WF}$.

(28) Suppose that M is an inner model of V, i.e., the assumptions of Example (27) hold. Suppose that $\omega^\omega \cap V = \omega^\omega \cap M$. Prove the following:

  (a) $\aleph_1^M = \aleph_1^V$,
  (b) if $A \subseteq \omega^\omega$ such that $A \in M$, then $M \models$"A is projective" if and only if $V \models$"A is projective", and
  (c) if $A \subseteq \omega^\omega$ such that $A \in M$, then $M \models$"A is determined" if and only if $V \models$"A is determined".

(29) Presentation Example. Show that every weakly compact cardinal is inaccessible.

[*Hint.* Note that for any cardinal λ the set 2^λ with the lexicographic ordering cannot have any strictly increasing or decreasing sequences of length $> \lambda$.]

(30) Prove that if κ is measurable, $c : [\kappa]^2 \to 2$, and U is a κ-complete non-principal ultrafilter on κ (not necessarily normal), then you can construct a c-homogeneous set H of cardinality κ. Compare your proof to our proof from Lecture XVIII where the ultrafilter was assumed to be normal.

(31) Let F be any filter on κ. If $S \subseteq \kappa$, a function $f : S \to \kappa$ is called *regressive* if for all $\xi \neq 0$, $f(\xi) < \xi$. A set S is called F*-stationary* if for all $X \in F$, we have that $X \cap S \neq \varnothing$.

Show that the filter F is normal if and only if for every F-stationary set S and every regressive function $f : S \to \kappa$ there is an $\alpha \in \kappa$ such that $\{\sigma \in S\,;\, f(\sigma) = \alpha\}$ is F-stationary.

(32) Let U be a κ-complete ultrafilter on κ. Show that the following are equivalent:

  (i) U is normal,
  (ii) for every function $f : \kappa \to \kappa$, if $\{\xi < \kappa\,;\, f(\xi) < \xi\} \in U$, then there is some $\alpha < \kappa$ such that $\{\xi < \kappa\,;\, f(\xi) = \alpha\} \in U$.

(33) Prove *Rowbottom's Theorem*: If κ is measurable, U is a normal ultrafilter on κ, $\gamma < \kappa$, and for every $k \in \mathbb{N}$, we have $c_k : [\kappa]^{<\omega} \to \gamma$, then there is a set $H \in U$ which is n-c_k-homogeneous for all n and k.

(34) Marked Example. Let s and t be finite sequences of ordinals. We say $s <_{\mathrm{KB}} t$ if either $t \subsetneq s$ (*sic*!) or if i if the least number such that $s(i) \neq t(i)$, we have $s(i) < t(i)$. This order is called the *Kleene-Brouwer order*. Show that it is a total order on the class of finite sequences of ordinals and that if T is a tree on κ, then T is well-founded if and only if $(T, <_{\mathrm{KB}})$ is a wellorder.

(35) A set $A \subseteq (\omega^\omega)^n$ is called κ*-Suslin* if there is a tree T on $\kappa \times \omega^n$ such that $A = \mathrm{p}[T]$. Prove that every set A is $2^{\aleph_0}$-Suslin.

(36) Prove that the class of κ-Suslin sets (cf. Example (35)) is closed under projections, continuous preimages and unions of size κ.

(37) Presentation Example. Let $A \subseteq \mathrm{WF} \times \mathrm{WF}$. The following game $\mathrm{G_S}(A)$ is called the *Solovay game* on A: players I and II produce a play z in the usual way, $x := z_{\mathrm{I}}$ and $y := z_{\mathrm{II}}$. Player I loses if $x \notin \mathrm{WF}$. Otherwise, player II loses if $y \notin \mathrm{WF}$. If both of them play in WF, then player I wins if $(x, y) \in A$.

Let $A := \{(x, y)\,;\, \|x\| \geq \|y\|\}$ and show that player I cannot have a winning strategy in $\mathrm{G_S}(A)$.

(38) Let σ be a winning strategy for player I in some Solovay game $\mathrm{G_S}(A)$. Show that there is a function $f : \aleph_1 \to \aleph_1$ such that if $\alpha = \|y\| < \xi$, then there is some x such that $\|x\| < f(\xi)$ and $(x, y) \in A$.